

SIXTH SEMESTER B.A./B.Sc. DEGREE EXAMINATION, MARCH 2020

(CUCBCSS-UG)

Mathematics

MAT 6B 09—REAL ANALYSIS

Time : Three Hours

Maximum : 120 Marks

Section A

*Answer all questions.**Each carries 1 mark.*

1. Define absolute maximum of a real valued function $f : A \rightarrow \mathbb{R}$, $A \subseteq \mathbb{R}$.
2. Give an example of a continuous function. Why unbounded eventhough its domain is a bounded set in \mathbb{R} .
3. State the location of Roots theorem.
4. Define uniform continuity of a function.
5. Find $\|p\|$ if $p = \{0, 0.2, 0.5, 0.9, 1.5, 2\}$ is a partition of $[0, 2]$.
6. Let F, G be differentiable on $[a, b]$ and $F' = f$ and $G' = g$ both belongs to $\mathbb{R}[a, b]$.
Then $\int_a^b fG = \text{_____}$.
7. Give an example of the improper integral of the 3rd kind.
8. The radius of convergence of the power series $\sum \frac{x^n}{n}$ is _____.
9. State the Weierstrass M-test for the uniform convergence of a series of functions.
10. When do we say a series of functions is absolutely convergent ?
11. $\lim_{n \rightarrow \infty} \left[\frac{\cos(nx + n)}{n} \right] = \text{_____}$.
12. State the relation between Beta function and Gamma function.

(12 × 1 = 12 marks)

Turn over

Section B

Answer any **ten** questions.
Each carries 4 marks.

13. (a) What do you mean by Lipschitz condition ?
(b) What is its geometrical interpretation ?
14. Let $f_n(x) = x^n(1-x)$, $x \in A = [0, 1]$. Prove that $f_n(x)$ converges to '0' uniformly on A.
15. Discuss the convergence of $\int_1^{\infty} \frac{\ln x \, dx}{x+u}$, where $u > 0$.
16. Evaluate $\int_1^4 \frac{\sin \sqrt{t}}{\sqrt{t}} dt$.
17. State the boundedness theorem on Riemann integral. Justify the converse by an example.
18. State continuous extension theorem. Use it to show that $f(x) = x \sin\left(\frac{1}{x}\right)$ is uniformly continuous on $(0, b]$, $b > 0$.
19. Show by an example that every uniformly continuous function need not be Lipschitz function.
20. Determine the uniform convergence of $\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$ for $|x| \leq a$.
21. Distinguish between pointwise and uniform convergence of a sequence. Write the relation between them. Also write the necessary and sufficient condition for sequence (f_n) fail to converge uniformly on $A_0 \subset \mathbb{R}$ to f .
22. If $f \in R[a, b]$, then P.T. $|f| \in R[a, b]$.
23. State the limit comparison test for the convergence of improper integrals. Test the convergence

of $\int_1^{\infty} \frac{dx}{1+x^2}$.

24. Prove that $\frac{1}{2} = \sqrt{\pi}$.

25. Prove that $\beta(m, n) = \beta(m+1, n) + \beta(m, n+1)$, $m, n > 0$.

26. Prove that $\beta(m, n) = 2 \int_0^{\pi/2} (\sin x)^{2m-1} (\cos x)^{2n-1} dx$.

(10 × 4 = 40 marks)

Section C

Answer any **six** questions.
Each carries 7 marks.

27. P.T. the image of a closed and bounded interval under a continuous mapping is a closed and bounded interval.
28. State and prove Bolzano's intermediate value theorem.
29. If $f: [a, b] \rightarrow \mathbb{R}$ is continuous $[a, b]$, then P.T. $f \in \mathbb{R}_{[a, b]}$.
30. State and prove uniform continuity theorem.
31. P.T. a sequence of bounded functions (f_n) on $A \subseteq \mathbb{R}$ converges uniformly to f on A iff $\|f_n - f\|_A \rightarrow 0$.
32. Discuss the convergence of $(f_n(x))$, where $f_n(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{N}$.
33. Prove that the uniform limit of a sequence of continuous function on $A \subseteq \mathbb{R}$ is continuous on A .
34. Prove that $\frac{\beta(m, n+1)}{n} = \frac{\beta(m+1, n)}{m} = \frac{\beta(m, n)}{m+n}$.
35. Evaluate $\int_0^1 (x \ln x)^3 dx$.

(6 × 7 = 42 marks)

Turn over

Section D

Answer any two questions.
Each carries 13 marks.

36. (a) State and prove Maximum-Minimum theorem.
(b) Show by an example that none of the conditions of the Maximum-Minimum theorem can be relaxed.
37. (a) If (x_n) is a Cauchy sequence on $A \subseteq \mathbb{R}$, then P.T. $(f(x_n))$ is a Cauchy sequence in \mathbb{R} , when $f: A \rightarrow \mathbb{R}$ is uniformly continuous on $A \subseteq \mathbb{R}$.
(b) Show that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

38. (a) Evaluate $\int_0^{\pi/2} \sqrt{\tan \theta} d\theta$.

(b) If $I = \int_0^{\infty} \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi}$, S.T. $n|1-n = \frac{\pi}{\sin n\pi}$ for $0 < n < 1$.

(2 × 13 = 26 marks)