D 51446 (Pages : 3) Name.....................................

Reg. No.....................................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2018

(CUCSS–PG)

Mathematics

MT 1C 04—NUMBER THEORY

(2016 Syllabus Year)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*

Each question carries a weightage of 1.

1. Prove that the Euler totient function $\phi(n)$ is even for $n \geq 3$.

2. Prove that the equation $f(n) = \sum_{d|n} g(d)$ implies $g(n) = \sum_{d|n} f(d) \cdot \mu\left(\frac{n}{d}\right)$.

3. Define the derivative of an arithmetical function. Find the derivative of the identity function.

4. State Euler's summation formula.

5. For $x \geq 1$, prove that :

$$\sum_{n \leq x} \wedge(n) \left[\frac{x}{n}\right] = \log[x]!$$

6. Prove that for the Chebyshev's functions $\psi(x)$ and $\vartheta(x)$, :

$$\psi(x) = \sum_{m \leq \log_2 x} \vartheta\left(x^{1/m}\right).$$

7. For $x \geq 2$, prove that :

$$\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t \log^2 t}\, dt.$$

8. For all $x \geq 1$, prove that :

$$\sum_{n \leq x} \vartheta\left(\frac{x}{n}\right) = x \log x + 0(x).$$

Turn over

9. For $n \geq 1$, prove that the n^{th} prime p_n satisfy the inequality $\frac{1}{6} n \log n < p_n$.

10. Find the quadratic residues and non-residues modulo II.

11. For every odd prime p, prove that :

$$(2 \mid p) = (-1)^{(p^2-1)/8}.$$

12. Evaluate the Jacobi symbol $(2 \mid 35)$.

13. How many different affine enciphering transformations are there with an N-letter alphabet ?

14. What are the disadvantages of deterministic encryption ?

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question carries a weightage of 2.

15. If $n \geq 1$, prove that :

$$\phi(n) = \sum_{d \mid n} \mu(d) \cdot \frac{n}{d}.$$

16. Let f be multiplicative. Prove that f is completely multiplicative if and only if :

$$f^{-1}(n) = \mu(n) f(n) \text{ for all } n \geq 1.$$

17. If $x \geq 2$, using Euler's summation formula, prove that :

$$\sum_{n \leq x} \frac{\log n}{n} = \frac{1}{2} \log^2 x + \mathrm{A} + \mathrm{O}\left(\frac{\log x}{x}\right)$$

where A is a constant.

18. State and prove Legendre's identity.

19. Prove that for every integer $n > 1$, there exist n consecutive composite numbers.

20. Prove that the n^{th} prime p_n satisfy the inequality :

$$p_n < 12\left(n \log n + n \log \frac{12}{e}\right), \text{ where } n \geq 1,$$

21. Prove that there is a constant A such that :

$$\sum_{p \leq n} \frac{1}{p} = \log \log x + \mathrm{A} + \mathrm{O}\left(\frac{1}{\log x}\right) \text{ for all } x \geq 2.$$

22. Prove that the Legendre's symbol $(n \mid p)$ is a completely multiplicative function of n.

23. Explain briefly about digraph transformations.

24. Briefly describe public key cryptosystem.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 4.

25. Show that the set of all multiplicative functions is a subgroup of the group of all arithmetical functions f with $f(1) \neq 0$ under dirichlet product.

26. Prove that the following relations are logically equivalent :

(a) $\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1.$

(b) $\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1.$

(c) $\lim_{x \to \infty} \frac{\psi(x)}{x} = 1.$

27. State and prove Quadratic reciprocity law.

28. The message " ! IWGVIEX!ZRADRYD " was intercepted. The message was sent using a linear enciphering transformation of digraph-vectors in a 29-letter alphabet in which A – Z have numerical equivalents 0 – 25, blank = 26, ? = 27, ! = 28. It is known that the last five letters of plaintext are the sender's signature "MARIA". Find the deciphering matrix and read the message.

(2 × 4 = 8 weightage)