

C 42026

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Name.....

Reg. No.....

**FOURTH SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY)
EXAMINATION, APRIL 2023**

(CBCSS)

Mathematics

MTH4E08—COMMUTATIVE ALGEBRA

(2019 Admission onwards)

Time : Three Hours

Maximum Weightage : 30

Part A

*Answer all questions.
Each question has weightage 1.*

1. Find all units in the ring \mathbb{Z}_6 of integers mod 6.
2. Find the nilradical of the ring \mathbb{Z}_{12} of integers mod 12.
3. Let \mathbb{Z}_{10} be an \mathbb{Z} -module under the usual action. Verify whether $N = \{0, 4, 8\}$ is a submodule of \mathbb{Z}_{10} .
4. Let $S^{-1}A$ be a ring of fractions and $f : A \rightarrow S^{-1}A$ be the natural inclusion homomorphism. Show that for some $a \in A$ if $f(a) = 0$ then $as = 0$ for some $s \in S$.
5. Let \mathbb{Z} be the ring of integers and $S = \{3^n : n \geq 0\}$. Verify whether $4/6$ is in $S^{-1}\mathbb{Z}$.
6. Give a factorizations of 6 into irreducibles in $\mathbb{Z}(\sqrt{-5})$ other than $6 = 3 \times 2$.
7. Verify whether $\sqrt{2}$ is integral over \mathbb{Z} in the field \mathbb{R} of reals.
8. Show that the ring \mathbb{Z} of integers doesnot have d.c.c. on ideals.

(8 × 1 = 8 weightage)

Part B

*Answer any two questions from each unit.
Each question has weightage 2.*

Unit I

9. Show that every nonzero prime ideal in a principal ideal domain is a maximal ideal.
10. Let M, N be A -modules and $f : M \rightarrow N$ be an A -module homomorphism. Show that $\ker f$ is a submodule of M and $\text{Im } f$ is a submodule of N .
11. Show that $0 \rightarrow M' \xrightarrow{f} M$ is an exact sequence if and only if f is injective.

Turn over

Unit II

12. Let \mathbb{Z} be the ring of integers and $S = \{p^n : n \geq 0\}$ where p is a prime. Find all units in $S^{-1}\mathbb{Z}$.
13. Let M, N be A -modules and $\phi : M \rightarrow N$ be an A -module homomorphism. Show that if ϕ is injective then $\phi_p : M_p \rightarrow N_p$ is injective for all prime ideals P of A .
14. Let q be a p -primary ideal in a ring A . Show that if $x \notin q$ then $r(q : x) = p$.

Unit III

15. Let A be a subring of a ring B such that B is integral over A . Let b be an ideal of B and $a = b \cap A$. Show that B/b is integral over A/a .
16. Let A be a subring of B and $C \subseteq B$ be the integral closure of A in B and S be a multiplicatively closed subset of A . Show that $S^{-1}C$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.
17. Let $0 \rightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \rightarrow 0$ be an exact sequence of A -modules. Show that if M is Artinian then M' and M'' are Artinian.

(6 × 2 = 12 weightage)

Part C

Answer any **two** questions.
Each question has weightage 5.

18. (a) Define local ring.
(b) Verify whether \mathbb{Z}_8 is a local ring.
(c) Let A be a ring and m be a maximal ideal of A such that every element of $A - m$ is a unit. Show that A is a local ring.
19. (a) Define flat A -module.
(b) Show that the following are equivalent for an A -module N .
(i) N is flat.
(ii) If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be any exact sequence of A -modules then the tensored sequence.
 $0 \rightarrow M' \otimes N \rightarrow M \otimes N \rightarrow M'' \otimes N \rightarrow 0$ is also exact.

20. (a) Let S be a multiplicatively closed subset of a ring A and q be a p -primary ideal of A . Show that :
- if $S \cap p \neq \emptyset$ then $S^{-1}q = S^{-1}A$.
 - if $S \cap p = \emptyset$ then $S^{-1}q$ is $S^{-1}p$ -primary.
- (b) Let q_1, q_2 be p -primary ideals of a ring A . Show that $q_1 \cap q_2$ is also p -primary.
21. (a) Define Noetherian module.
- (b) Show that if k is a field then the polynomial ring $k[x]$ is Noetherian.
- (c) Let M be an A -module. Prove that M is Noetherian if and only if M is finitely generated.
- (2 × 5 = 10 weightage)