

SIXTH SEMESTER B.A./B.Sc. DEGREE EXAMINATION, MARCH 2020

(CUCBCSS—UG)

Mathematics

MAT 6B 12—NUMBER THEORY AND LINEAR ALGEBRA

Time : Three Hours

Maximum : 120 Marks

Section A

*Answer all the twelve questions.**Each question carries 1 mark.*

1. State Fundamental theorem of Algebra.
2. State division algorithm for integers.
3. Find a value of c for which the Diophantine equation $24x + 36y = c$ has a solution.
4. Write the number of solutions of the linear congruence $18x = 30 \pmod{42}$, which are incongruent modulo 42.
5. State Fermat's Little Theorem.
6. Find $\sigma(12)$.
7. Find $\phi(30)$.
8. What do you mean by a subspace of a vector space ?
9. Define basis of a vector space.
10. Define a linear map.
11. If the mapping $f: V \rightarrow W$ is linear then show that $f(0_v) = 0_w$.
12. Define the nullity of a linear map.

(12 × 1 = 12 marks)

Section B

*Answer any ten questions from among the questions 13 to 26.**Each question carries 4 marks.*

13. Show that any integer of the form $6k + 5$ is also of the form $3j + 2$ but not conversely.
14. Prove that $\gcd(ka, kb) = k \gcd(a, b)$ where k is any positive integer.

Turn over

15. Find one integer solutions of the Diophantine equation $33x + 14y = 115$.
16. If p is a prime number and $p|ab$, then show that $p|a$ or $p|b$.
17. Prove that $\sqrt{5}$ is irrational.
18. Without performing the division, determine whether the integer 176521221 is divisible by 9 or 11.
19. Using Fermat's theorem, show that 17 divides $11^{104} + 1$.
20. Show that the function τ is multiplicative.
21. Given integers a, b, c , then prove that $\gcd(a, bc) = 1$ if and only if $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$.
22. Show that the intersection of any set of subspaces of a vector space V is a subspace of V .
23. If V is a vector space over C of dimension n , prove that V can be regarded as a vector space over R of dimension $2n$.
24. Let V be a vector space of dimension $n \geq 1$ over a field F . Then show that V is isomorphic to the vector space F^n .
25. If $f : R^2 \rightarrow R^2$ is given by $f(a, b) = (b, 0)$, prove that $Imf = Kerf$.
26. Let $A = \{(x, y, 0); x, y \in R\}$ and let $B = \{(x, 0, z); x, y, z \in R\}$, subspaces of R^3 . Show that $A = B$.

(10 × 4 = 40 marks)

Section C

Answer any **six** questions from among the questions 27 to 35.

Each question carries 7 marks.

27. Given integers a and b with $b > 0$, prove that there exist unique integers q and r satisfying $a = bq + r, 0 \leq r < b$.
28. Suppose a and b are integers, not both zero. For a positive integer d , prove that, $d = \gcd(a, b)$ if and only if (a) $d|a$ and $d|b$; (b) Whenever $c|a$ and $c|b$ then $c|d$.
29. If $p \geq q \geq 5$ and p and q are both primes, prove that $24|p^2 - q^2$.
30. State and prove Euclid theorem on primes.
31. Derive Legendre formula for $n!$

32. Show that a non-empty subset S of a vector space V is a basis of V if and only if every element of V can be expressed in a unique way as a linear combination of elements of S .
33. Show that $(1, 1, 0, 0), (-1, -1, 1, 2), (1, -1, 1, 3), (0, 1, -1, -3)$ is a basis of \mathbb{R}^4 and express a general vector (a, b, c, d) as a linear combination of these basis elements.
34. Let $f: V \rightarrow W$ be linear. If X is a subspace of V then show that $f^{-1}(X)$ is a subspace of V ; and if Y is a subspace of W then show that $f^{-1}(Y)$ is a subspace of V .
35. Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be linear such that $f(1, 1, 0) = (1, 2), f(1, 0, 1) = (0, 0), f(0, 1, 1) = (2, 1)$. Determine f completely.

(6 × 7 = 42 marks)

Section D

Answer any two questions from among the questions 36 to 38.

Each question carries 13 marks.

36. Prove that every positive integer $n > 1$ can be expressed as product of primes ; this representation is unique apart from the order in which factors occur.
37. State and prove Chinese Remainder Theorem.
38. Let V and W be vector spaces of finite dimension over a field F . If $f: V \rightarrow W$ is linear then show that $\dim V = \dim \text{Im} f + \dim \text{Ker} f$.

(2 × 13 = 26 marks)