

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, MARCH 2020

(CUCSS)

Mathematics

MT 4E 02—ALGEBRAIC NUMBER THEORY

Time : Three Hours

Maximum : 36 Weightage

Part A

*Answer all questions.**Each question carries weightage 1.*

1. Let H be a subgroup of a free abelian groups G of rank r . Show that $\frac{G}{H}$ is finite iff the ranks of G and H are equal.
2. Express $\mathbb{Q}(\sqrt{2}, \sqrt[3]{5})$ in the form $\mathbb{Q}(\theta)$.
3. Find all monomorphisms $\sigma : K \rightarrow \mathbb{C}$, where $K = \mathbb{Q}(\sqrt[4]{2})$.
4. Show that if $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is any \mathbb{Q} -basis of a number field K , then
$$\Delta[\alpha_1, \alpha_2, \dots, \alpha_n] = \det(T(\alpha_i, \alpha_j)).$$
5. Show that the quadratic fields are precisely those of the form $\mathbb{Q}(\sqrt{d})$ for a square free rational integer d .
6. Find integral basis and discriminant for $\mathbb{Q}(\sqrt{3})$.
7. Let D be the ring of integers in a number field K and let $x \in D$. Show that x is a unit if and only if $N(x) = \pm 1$.
8. Show that the ring of integers D in a number field K is noetherian.
9. Verify that the element $3 - 7i$ of $\mathbb{Z}[i]$ is irreducible.

Turn over

10. Show that the ideal $p = \langle 3, 1 - \sqrt{5} \rangle$ of $Z[\sqrt{-5}]$ is prime.
11. Let L be an m -dimensional lattice in R^n . Show that $\frac{R^n}{L}$ is isomorphic to $T^m \times R^{n-m}$.
12. Determine the σ -map if $K = \mathbb{Q}(\theta)$ where $\theta \in R$ and $\theta^3 = 1$.
13. Find all squarefree integers d in $-10 < d < 10$ such that the class-number of $\mathbb{Q}(\sqrt{d})$ is 1.
14. Factorize the principle ideal $\langle 5 \rangle$ in the ring D of integers of the number field $K = \mathbb{Q}(\sqrt{3})$.

(14 × 1 = 14 weightage)

Part B

*Answer any seven questions.
Each question carries weightage 2.*

15. Show that the discriminant of any basis for $K = \mathbb{Q}(\theta)$ is rational and non-zero.
16. Show that a complex number θ is an algebraic integer if and only if the additive group generated by all powers $1, \theta, \theta^2, \dots$ is finitely generated.
17. Show that every number field K of degree n possesses an integral basis, and the additive group of the ring of integers D is free abelian of rank n equal to the degree of K .
18. Show that the discriminant of $\mathbb{Q}(\xi)$, where $\xi = e^{\frac{2\pi i}{p}}$ and p is an odd prime, is $(-1)^{\frac{p-1}{2}} \cdot p^{p-2}$.
19. Show that if a domain D is noetherian, then factorization into irreducibles is possible in D .
20. Show that factorization into irreducibles is not unique in the ring of integers of $\mathbb{Q}(\sqrt{-13})$.
21. Suppose p, q are distinct prime ideals in the ring of integers D of a number field K of degree n . Show that $p + q = D$ and $p \cap q = pq$.

22. Show that if X is a bounded subset of \mathbb{R}^n and $v(X)$ exists and if $v(\gamma(x)) = v(x)$, then γ/x is not injective, where $\gamma: \mathbb{R}^n \rightarrow \mathbb{T}^n$ is the natural homomorphism.
23. Determine the class-number of the ring of integers of $\mathbb{Q}(\sqrt{-5})$.
24. Let $K = \mathbb{Q}(\xi)$, where $\xi = e^{\frac{2\pi i}{p}}$ for an odd prime p . Show that the only roots of unity in K are $\pm \xi^s$ for integers s .

(7 × 2 = 14 weightage)

Part C

*Answer any two questions.
Each question carries weightage 4.*

25. Show that every subgroup H of a free abelian group G of rank n is of rank $s \leq n$.
26. Show that the ring of integers D of $\mathbb{Q}(\sqrt{-3})$ is Euclidean.
27. Let D be the ring of integers of a number field K of degree n . Show that if a and b are non-zero ideals of D , then $N(ab) = N(a)N(b)$.
28. Show that the solutions of $x^2 + y^2 = z^2$ with pairwise coprime integers x, y, z are given parametrically by $\pm x = r^2 - s^2, \pm y = 2rs, \pm z = r^2 + s^2$.

(2 × 4 = 8 weightage)