92413

C 20207 (Pages : 3) Name.................................

Reg. No.............................

SIXTH SEMESTER (CUCBCSS–UG) DEGREE EXAMINATION, MARCH 2022

Mathematics

MAT 6B 09—REAL ANALYSIS

(2014 to 2018 Admissions)

Time : Three Hours Maximum : 120 Marks

Part A

Answer **all** *questions.*
Each question carries 1 mark.

1. Define uniform continuity of a function.
2. State Weierstrass approximation theorem.
3. Find $\|P\|$ if $P = \{0, 2, 3, 4\}$ in a partition of $[0, 4]$.
4. Give an example for a function which is not Riemann integrable.
5. Define step function.
6. State Lebesgue integrability criterion.
7. Define uniform convergence of a series of functions.
8. $\lim_{n\to\infty} \frac{x^2 + nx}{n}$.
9. Write an example for an absolutely convergent improper integral.
10. Cauchy principal value of $\int_{-\infty}^{\infty} x\,dx =$.
11. Define Beta function.
12. Fill in the blanks : $\Gamma(3) =$ ———.

(12 × 1 = 12 marks)

Part B

Answer any **ten** *questions.*
Each question carries 4 marks.

13. Let $I = [a, b]$ be a closed boudned interval and $f : I \to \mathbb{R}$ be continuous on I. If $k \in \mathbb{R}$ is any number satisfying $\inf f(I) \le k \le \sup f(I)$ then prove that there exists a number $c \in I$ such that $f(c) = k$.
14. State and prove Preservation of Intervals theorem.

Turn over

15. Show by an example that every uniformly continuous function need not be a Lipschitz function.

16. If $\phi : [a,b] \to \mathbb{R}$ is a step function, prove that $\phi \in \mathfrak{R}[a,b]$.

17. Suppose that $f, g \in \mathfrak{R}[a,b]$. Prove that $fg \in \mathfrak{R}[a,b]$.

18. State the substitution theorem of Riemann integration. Use it to evaluate $\int_0^4 \frac{\sin\sqrt{t}}{\sqrt{t}}\, dt$.

19. Let (f_n) be a sequence of bounded functions of $A \subseteq \mathbb{R}$. Suppose that $\|f_n - f\|_A \to 0$. Then prove that (f_n) converges uniformy on A to f.

20. If f_n is continuous of $D \subseteq \mathbb{R}$ to $\mathbb{R}$ for each $n \in \mathbb{N}$ and if $\sum f_n$ converges to f uniformly on D, then prove that f is continuous on D.

21. State and prove Weierstrass M-Test for a series of functions.

22. Discuss the uniform convergence of $\sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}$.

23. Test the convergence of $\int_0^{\infty} \frac{1}{x^2}\, dx$.

24. Show that $\Gamma(n+1) = n!$ when n is a positive integer.

25. Show that $\beta(m, n) = \beta(n, m)$.

26. Evaluate $\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx$.

(10 × 4 = 40 marks)

Part C

Answer any **six** *questions.*
Each question carries 7 marks.

27. Let $I = [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then prove that f is bounded on I.

28. State and prove Uniform Continuity Theorem.

29. State and prove Continuous Extension Theorem.

30. If $f : [a, b] \to \mathbb{R}$ is monotone on $[a, b]$ then prove that $f \in \mathfrak{R}[a,b]$.

31. Discuss the convergence of the sequence $(f_n(x))$ where $f_n(x) = \frac{x^n}{x^n + 1}$, $x \in [0, 2]$.

32. State and prove Taylor's Theorem with the Reminder.

33. Let $f \in \mathfrak{R}[a,b]$ and let f be continuous at a point $c \in [a,b]$. Prove that the indefinite integral $F(z) = \int_a^z f$ foe $z \in [a,b]$ is differentiable at c and $F'(c) = f(c)$.

34. Show that $\Gamma\left(\frac{P}{2}\right)\Gamma\left(\frac{P+1}{2}\right) = \frac{\sqrt{\pi}}{2^{P-1}}\Gamma(p)$.

35. Evaluate the integral $\int_0^1 x^2\left(1-\sqrt{x}\right)dx$.

(6 × 7 = 42 marks)

Part D

Answer any **two** *questions.*
Each question carries 13 *marks.*

36. (a) State and prove Maximum Minimum Theorem.

    (b) Test the uniform continuity of $f(x) = \sqrt{x}$ on [0, 2].

37. (a) Let $f:[a,b] \to \mathbb{R}$ and $c \in (a,b)$. Prove that $f \in \mathfrak{R}[a,b]$ if and only if its restriction to $[a, c]$ and $[c, b]$ are both Riemann integrable. In this case show that $\int_a^b f = \int_a^c f + \int_c^b f$.

    (b) If $f \in \mathfrak{R}[a,b]$ and if $[c,d] \subseteq [a,b]$ then prove that the restriction of f to $[c, d]$ is in $\mathfrak{R}[c, d]$.

38. (a) Prove that $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \forall m, n > 0$.

    (b) Find the value of $\Gamma\left(\frac{1}{2}\right)$.

(2 × 13 = 26 marks)