D 72784 (Pages : 4) Name..............................

Reg. No..............................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2019

(CUCSS)

Mathematics

MT1C03—REAL ANALYSIS—I

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A (Short Answer Questions)

Answer **all** *questions.*
Each question carries weightage 1.

1. Let E be a dense subset of a metric space and $f : X \to Y$ be a continuous function. Prove that f(E) is dense in f(X).

2. Show that the set of discontinuities of a monotonic function $f : (a, b) \to \mathbb{R}$ is at most countable.

3. Prove that the closure $\bar{E}$ of a subset E of a metric space X is closed in X.

4. Show that if E is an infinite subset of a compact metric space K, then E has a limit point in K.

5. Show that a subset E of a metric space is open if and only if its complement of closed.

6. Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. Prove or disprove that f' is continuous.

7. Define the Riemann integral for a bounded real function on $[a, b]$.

8. If

$$f(x) = \begin{cases} 1 & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ rational} \end{cases}$$

prove that $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$.

9. Show that for every continuous function $f : [-1, 1] \to [0, 1]$ such that $f(0) < 0 < f(1)$, there is point $x \in [0, 1]$ such that $f(x) = 0$.

Turn over

10. Show that the function given by :

$$f(x)=\begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at every $x \in \mathbb{R}$.

11. Verify whether the sequence $\{f_n'\}$ converge to f', where $f_n(x)=\frac{\sin nx}{\sqrt{n}}$ and $f(x)=\lim_{n\to\infty} f_n(x)$.

12. Verify whether the sequence of functions $\{f_n\}$ given by $f_n(x)=\frac{x^2}{x^2+(1-nx)^2}$, $n = 1, 2, 3 \ldots$ has a uniformly convergent subsequence.

13. Evaluate the sum of functions $\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$ on $\mathbb{R}$.

14. Verify whether the sequence of functions $f_n(x)=\frac{1}{1+nx}$ on $(0, 1)$ $n = 1, 2, 3, \ldots$, converge uniformly on $(0, 1)$.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question carries weightage 2.

15. Define :

$$d(p, q)=\begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}$$

on an infinite set X. Prove that d is a metric. Which subsets of this metric space are compact. Justify your answer.

16. Prove that if E is a non-empty bounded above subset of $\mathbb{R}$ and $y := \sup \text{E}$, then $y \in \bar{\text{E}}$.

17. Let $E \subset \mathbb{R}$ be a non-compact subset. Show that there is a continuous function on E which is not bounded.

18. If f is a continuous function on $[a, b]$ into $\mathbb{R}^k$ and f is differentiable in (a, b), prove that there exists an $x \in (a, b)$ such that $|f(a) - f(b)| \le (b - a)|f'(x)|$.

19. State L' Hospital's rule for real functions f and g defined and differentiable on (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$. Show by an example that the result may fail to be true for complex valued functions.

20. If f is monotonic and α is monotonically increasing and continuous on $[a, b]$, prove that $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

21. Let f and F be maps from $[a, b]$ into $\mathbb{R}^k$, $f \in \mathcal{R}$ on $[a, b]$ and $F' = f$. Prove that $\int_a^b f(t)\,dt = F(b) - F(a)$.

22. State and prove Weierstrass test for uniform convergence of series of real valued functions defined on a set E of a metric space.

23. State and prove the Cauchy criterion for uniform convergence of sequence of real functions on a subset E of a metric space.

24. Prove that a sequence $\{f_n\}$ converges to f with respect to the supremum metric of C (X) if and only if $f_n \to f$ uniformly on X.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries weightage 4.

25. (a) Let

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f has a discontinuity of the second kind at $x = 0$.

(b) Give an example of a function having exactly countably many simple discontinuities.

26. Show that any non-empty perfect set in $\mathbb{R}^*$ is uncountable.

27. Show that $f \in \mathcal{R}(\alpha)$ on $[a, b]$ if and only if for any $\epsilon > 0$ there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

28. Let $\{f_n\}$ be a sequence of continuous functions on a compact metric space K. If $\{f_n\}$ is uniformly bounded and equicontinuous on K, prove that it has a uniformly convergent subsequence.

(2 × 4 = 8 weightage)