C 80707 (Pages : 3) Name..............................

Reg. No..............................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, MARCH 2020

(CUCSS)

Mathematics

MT4 E06—ADVANCED COMPLEX ANALYSIS

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*

Each question carries a weightage of 1.

1. Construct a meromorphic function that has a simple pole at $z = n^2$ with residue n for every $n \in \mathbb{N}$.
2. Prove that every meromorphic function can be expressed as the quotient of two entire functions.
3. Prove that the infinite product $\prod_{k=1}^{\infty}(1+C_k)$ converges absolutely if and only if the infinite series $\sum_{k=1}^{\infty} C_k$ converges absolutely.
4. Discuss the convergence of the infinite product $\prod_{k=2}^{\infty} \frac{k^2-1}{k^2}$.
5. Show that $f(z) = \prod_{k=1}^{\infty}\left(1-\pi z^{k^2}\right)$ define an analytic function in the unit disc D.
6. Find the order of the entire function $f(z) = ze^z$.
7. Prove that a non-constant entire function of finite order assumes every complex value with only one possible exception.
8. Show that $\frac{1}{1-i}\sum_{k=0}^{\infty}\left(\frac{z-i}{1-i}\right)^k$ is a direct analytic continuation of $\sum_{k=0}^{\infty} z^k$.
9. Give an example of a power series whose circle of convergence is its natural boundary.
10. State the Monodromy theorem.

Turn over

11. If G is an open set in $\mathbb{C}$, then prove that there is a sequence $\{k_n\}$ of compact subsets of G such that

$$G = \bigcup_{n=1}^{\infty} k_n.$$

12. Let $\mathcal{F} \subset C(G,\Omega)$ be normal. Prove that $\overline{\mathcal{F}}$ is compact.

13. If $\{f_n\}$ is a sequence in H (G) (the space of analytic functions on G) and if f belongs to C (G, $\mathbb{C}$) such that $f_n \to f$, then prove that f is analytic.

14. Show that the space of meromorphic functions M (G) is not complete.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions.*

Each question carries a weightage of 2.

15. Prove that the infinite product $\prod_{k=1}^{\infty} a_k$ converges if and only if for every $\epsilon > 0$, there is an integer N such that $\left|\left(\prod_{k=n+1}^{m} a_k\right) - 1\right| < \epsilon$ whenever $m > n \geq N$.

16. If $Z \in \overline{D}(0,1)$, prove that $|1 - E_n(z)| \leq |z|^{n+1}$

17. Let $f(z)$ be a bounded analytic function, not identically zero, in the unit disc D. If $\{a_n\}_{n=1}^{\infty}$ is the sequence of roots of $f(z)$ in the unit disc D, each repeated according to its multiplicities, then prove that the product $\prod_{n=1}^{\infty} |a_n|$ is convergent.

18. Construct an entire function $f(z)$ such that $f(n) = n!(n = 0, 1, 2, ...)$.

19. Suppose $f(z)$ is a function continuous on Γ, where Γ consists of a finite number of curves $C_k\,(k = 1, 2, ..., m)$. Let K be a compact set that does not intersect Γ. Prove that for any $\epsilon > 0$, there exists a rational function $\varphi(z)$ having all its poles on Γ and satisfying $\left|\frac{1}{2\pi i}\int_{\Gamma}\frac{f(g)}{g-z}\,d\,g - 4(z)\right| < \epsilon$ for all $z \in k$.

20. Let $f(z)$ be a power series centered at $z = a$ and let Γ be an arc joining a to b. If $f(z)$ can be continued analytically along the arc Γ, then prove that the analytic continuation of $f(z)$ at the point b along the arc Γ is unique.

21. Prove that $(C(G, \Omega), \rho)$ is a complete metric space.

22. Suppose $\mathcal{F} \subset C(G, \Omega)$ is equicontinuous at each point of G. Prove that $\mathcal{F}$ is equicontinuous over each compact subset of G.

23. State and prove Hurwitz's theorem.

24. Let G be a region which is not the whole plane and such that every non-vanishing analytic function on G has an analytic square root. If $a \in G$, prove that there is an analytic function f on G such that $f(a) = 0$ and $f'(a) > 0$.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 4.

25. Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of distinct points having no finite accumulation point and let a sequence of positive integers $\{k_j\}_{j=1}^{\infty}$ be given. Prove that there exists an entire function having roots of multiplicity k_j at a_j for all $j \in N$ and nowhere else.

26. State and prove Hadamard factorization theorem.

27. State and prove Schwarz symmetry principle.

28. Prove that a family $\mathcal{F}$ in H (G) is normal if and only if $\mathcal{F}$ is locally bounded.

(2 × 4 = 8 weightage)