C 31282 (Pages : 3) Name..................................

Reg. No..................................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2017

(CUCSS)

Mathematics

MT 3C 12—FUNCTIONAL ANALYSIS—I

(2010 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A (Short Answer Questions)

Answer **all** *questions.*
Each question carries 1 *weightage.*

1. Prove that L^{∞} is not separable.
2. Prove that discrete metric space are complete.
3. Prove that $\int_{-\pi}^{\pi} |D_n(t)| \, dt \to \infty$, as $n \to \infty$ where $D_n(t)$ is the nth Dirichlet kernel.
4. Prove that addition is continuous in a normed space.
5. Let Y be a subspace of a normal space X. Prove that $Y^0 \neq \phi$ if and only if Y = X.
6. Define strictly convex normed spaces. Prove that $\mathbb{R}^2$ with $\| \ \|_2$ is strictly convex.
7. Prove that the norms $\| \ \|_{\infty}$ and $\| \ \|_1$ on $\mathbb{C}^n$ are comparable.
8. Prove that the function f from the real normed space $\mathbb{R}^2$ with norm $\| \ \|_2$ to $\mathbb{R}$ defined by $f(x, y) = 2x + 3y$ is linear and continuous.
9. Let $\langle , \rangle$ be an inner product in the linear space X. For $x, y \in X$, prove that

$$\|x\|^2 + \|y\|^2 = \frac{1}{2}\left(\|x+y\|^2 + \|x-y\|^2\right),$$

where $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$.

10. Prove that every orthonormal set in an inner product space is linearly independent.
11. Let X be an inner product space, let $E \subset X$ and $x \in \overline{E}$. Prove that there is a best approximation from E to x if and only if $x \in E$.
12. State Hahn-Banach extension theorem.

Turn over

13. Prove that there exists continuous linear functional f on the normed space $\mathbb{R}^2$ with $\| \ \|_2$ such that $\|f\| = 1$.

14. Prove that in a Banach space X every absolutely summable series of elements in X is summable in X.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question carries 2 *weightage.*

15. Let E be a measurable subset of $\mathbb{R}$. If $m(E) < \infty$ and $1 \le p < \infty$, then prove that the set of all bounded continuous functions on E is dense in $L^p(E)$.

16. Let $x \in L^1([-\pi, \pi])$. If $\hat{x}(n) = 0$ for all $n = 0, \pm 1, \pm 2, \ldots$, then prove that $x(t) = 0$ for almost all $t \in [-\pi, \pi]$.

17. Prove that finite dimensional normed spaces are complete.

18. Let X and Y normed spaces and let $F : X \to Y$ be a linear map. Prove that F is continuous if and only if for every Cauchy sequence $\{x_n\}$ in X, the sequence $\{F(x_n)\}$ is Cauchy in Y.

19. Let X be a finite dimensional normed space and Y be a normed space. Prove that every bijective linear map from X onto Y is a homeomorphism.

20. Let $\{u_\alpha\}$ be an orthonormal basis for a Hilbert space H. Prove that

$$\|x\|^2 = \sum_n |\langle x, u_n \rangle|^2$$

where $x \in X$ and $\{u_1, u_2, \ldots\} = \{u_\alpha \, ; \langle x, u_\alpha \rangle \neq 0\}$.

21. Let E be a non-empty closed convex subset of a Hilbert space H. Prove that for each $x \in H$, there is a best approximation from E to x.

22. Let X be a normed space over a field K and let f be a non-zero linear functional on X. If E is an open subset of X, then prove that f (E) is an open subset of K.

23. Let X be a normed space over a field K and Y be a subspace of X. If $a \in X$ with $a \notin \overline{Y}$, then prove that there is a bounded linear functional f on X such that $f|_Y = 0$, $f(z) = \text{dist}(a, \overline{Y})$ and $\|f\| = 1$.

24. Let X and Y be normed space and let $F : X \to Y$ be a linear map. Prove that F is continuous if and only if $g \circ F$ is continuous for every continuous linear functional g on Y.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries 4 weightage.

25. Let X be a normed space. Prove that the following are equivalent :
    (i) Every closed and bounded subset of X is compact.
    (ii) The subset $\{x \in X : \|x\| \leq 1\}$ of X is compact.
    (iii) X is finite dimensional.
26. State and prove Bessel's inequality.
27. Let Y be a closed subspace of a normed space X. For $x + Y$ in the quotient space X/Y, let

$$|||x + Y||| = \inf\{\|x + y\| : y \in Y\}.$$

Prove that $||| \ |||$ is a norm on X/Y. Also prove that a sequence $(x_n + Y)$ converges to $x + Y$ in X/Y if and only if there is a sequence (y_n) in Y such that the sequence $(x_n + y_n)$ converges to x in X.

28. Let X, be a normed space over a field K, E be a non-empty open convex subset of X and Y be a subspace of X such that $E \cap Y = \phi$. Prove that there is a closed hyperspace Z in X such that $Y \subset Z$ and $E \cap Z = \phi$.

(2 × 4 = 8 weightage)