C 2044 (Pages : 3) Name..............................

Reg. No..............................

FOURTH SEMESTER M.Sc. DEGREE (REGULAR) EXAMINATION MARCH 2021

(CBCSS)

Mathematics

MTH 4C 15—ADVANCED FUNCTIONAL ANALYSIS

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *the questions.*
Each question carries a weightage of 1.

1. Show that every $\lambda \in \mathbb{C}$ with $|\lambda| > \|A\|$ is a regular point of the operator A.
2. Give an example for a residual spectrum.
3. Show that for every compact operator $T, 0 \in \sigma(T)$, the spectrum of the operator T.
4. Show that every ortho projection satisfies $0 \le P \le I$.
5. State the Banach-Steinhaus theorem.
6. Give an example of a set which is convex but is not perfectly convex.
7. Let E_1, E_2 be Banach spaces, $A \in L(E_1, E_2)$ and let $T_1 \subseteq E_1, T_1 \subseteq E_2$ be perfectly convex sets. Prove that if T_1 is bounded, then AT_1 is perfectly convex.
8. Give an example of a Banach algebra without identity.

(8 × 1 = 8 weightage)

Turn over

Part B

Answer any **six** *questions.*
Each question carries a weightage of 2.

Unit 1

9. Show that $\langle Ax,x\rangle \in \mathbb{R}$ for any $x \in H$ if and only if A is symmetric.

10. Define the operator $K : L_2[0,1] \mapsto L_2[0,1]$ by $(Kf)(t) = \int_0^1 k(t,s)\, f(s)\, ds,$ where $k(t,s) = \begin{cases} 1, & s \le t, \\ 0, & s > t. \end{cases}$
Find the spectrum of K.

11. Let A be a symmetric operator and let $\| A \| = \mu = \sup\{|\langle Ax,x\rangle| : \|x\|\} = 1$. Show that at least one of μ or $-\mu$ is an element of $\sigma(A)$.

Unit 2

12. Prove that for any self-adjoint operator $A \in L(H)$ the residual spectrum is empty.

13. Let $\varphi(t) \in K[a,b]$, the set of piece-wise continuous bounded functions which are monotone decreasing limits of continuous functions. Show that there exists a sequence of polynomials $P_n(t) \searrow \varphi(t)$ as $n \to \infty$ for all $t \in [a,b]$.

14. State the Hilbert theorem on the spectral decomposition of self- adjoint bounded operators.

Unit 3

15. Define closed graph operator and give an example for a closed graph operator.

16. If X^* is separable, then show that X is also separable.

17. Let $A : X \mapsto Y$ be a linear operator such that Im (A) is closed in Y and there exists $m > 0$ such that for any $x \in \text{Dom } A$, $\| Ax \| \ge m\| x \|$. Prove that A is closed.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. Show that a sequence of operators $T_n \in L(X,Y)$ converges strongly to an operator $T \in L(X,Y)$ if and only if :

    (i) the sequence $\{T_n(x)\}$ converges for any x from a dense subset of X.

    (ii) there exists $C > 0$ such that $\| T_n \| \leq C$.

19. State and prove the Gelfand's theorem on maximal ideals.

20. Let the operator $K : L_2[-\pi,\pi] \mapsto L_2[0,1]$ be given by $(Kf)(t) = \int_{-\pi}^{\pi} |t-s| f(s) ds$.

    (a) Prove that K is a compact self - adjoint operator.

    (b) Find the spectrum of K.

    (c) Is K a positive operator ? Justify.

21. State and prove the Fredholem's first theorem.

(2 × 5 = 10 weightage)