

C 42024

(Pages : 3)

Name.....

Reg. No.....

**FOURTH SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY)  
EXAMINATION, APRIL 2023**

(CBCSS)

Mathematics

MTH4E06—ALGEBRAIC NUMBER THEORY

(2019 Admission onwards)

Time : Three Hours

Maximum : 30 Weightage

**Part A**

*Answer all questions.  
Each question has weightage 1.*

1. Find the order of the group  $G/H$  where  $G$  is free abelian with  $\mathbb{Z}$ -basis  $x, y, z$  and  $H$  is generated by  $3x + y - 2z, 4x - 5y + z$  and  $x + 7z$ .
2. Let  $D$  be the ring of integers of a number field  $K$ . If  $\alpha \in K$ , then prove that  $c\alpha \in D$  for some non-zero  $c \in \mathbb{Z}$ .
3. Let  $K = \mathbb{Q}(\zeta)$ , where  $\zeta = e^{2\pi i/5}$ . Calculate  $T_K(\alpha)$  where  $\alpha = \zeta^2$ .
4. Prove that a prime in a domain  $D$  is irreducible.
5. Define a Euclidean domain.
6. Prove that the ring of integers  $D$  in a number field  $K$  is noetherian.
7. Sketch the fundamental domain for the lattice in  $\mathbb{R}^2$  generated by  $(1,1)$  and  $(2,3)$ .
8. Let  $K$  be a number field of class number  $h$  and  $\mathfrak{a}$  an ideal of the ring of integers  $D$  of  $K$ . Prove that  $\mathfrak{a}^h$  is principal.

(8 × 1 = 8 weightage)

Turn over

## Part B

Answer any **six** questions, by choosing two questions from each unit.  
Each question has weightage 2.

## Unit I

9. Prove that a subgroup of a finitely generated abelian group is finitely generated.
10. Let  $d$  be a squarefree rational integer. Prove that the integers of  $\mathbb{Q}(\sqrt{d})$  are  $\mathbb{Z}[\sqrt{d}]$  if  $d \not\equiv 1 \pmod{4}$  and  $\mathbb{Z}\left[\frac{1}{2} + \frac{1}{2}\sqrt{d}\right]$  if  $d \equiv 1 \pmod{4}$ .
11. Prove that the discriminant of any basis for a number field  $K = \mathbb{Q}(\theta)$  is rational and non-zero.

## Unit II

12. Prove that every principal ideal domain is a unique factorization domain.
13. If  $\mathfrak{a}$  and  $\mathfrak{b}$  are non-zero ideals of the ring of integers  $D$  in a number field, then prove that there exists an  $\alpha \in \mathfrak{a}$  such that :
- $$\alpha \mathfrak{a}^{-1} + \mathfrak{b} = D.$$
14. For squarefree  $d < -11$ , prove that the ring of integers  $\mathbb{Q}(\sqrt{d})$  is not Euclidean.

## Unit III

15. Let  $X$  be a bounded subset of  $\mathbb{R}^n$  and suppose that the volume  $v(X)$  exists. Let  $\nu : \mathbb{R}^n \rightarrow \mathbb{T}^n$  be the natural isomorphism with kernel  $L$ . If  $v(\nu(X)) \neq v(X)$ , then prove that  $\nu|_X$  is not injective.
16. Let  $D$  be the ring of integers of a number field  $K$  of degree  $n = s + 2t$  as usual, and let  $0 \neq \mathfrak{a}$  be an ideal of  $D$ . Prove that the volume of a fundamental domain for  $\sigma(\mathfrak{a})$  in  $\mathbb{L}^{st}$  is  $2^{-t} N(\mathfrak{a}) \sqrt{|\Delta|}$  where  $\Delta$  is the discriminant of  $K$ .
17. Prove that the equation  $x^4 + y^4 = z^2$  has no integer solutions with  $x, y, z \neq 0$ .

(6 × 2 = 12 weightage)

**Part C**

*Answer any two questions.  
Each question has weightage 5.*

18. (a) Prove that every number field possesses an integral basis.  
(b) Find the ring of integers of  $\mathbb{Q}(\sqrt[3]{175})$ .
19. Prove that the ring of integers of  $\mathbb{Q}(\zeta)$  is  $\mathbb{Z}(\zeta)$ , where  $\zeta = e^{2\pi i/p}$ ,  $p$  be an odd prime.
20. In a domain in which factorization into irreducibles is possible, prove that factorization is unique if and only if every irreducible is prime.
21. (a) Let  $K = \mathbb{Q}(\zeta)$ , where  $\zeta = e^{2\pi i/p}$ ,  $p$ , an odd prime. Prove that the only roots of unity in  $K$  are  $\pm \zeta^s$  for integers  $s$ .  
(b) If  $p(t) \in \mathbb{Z}[t]$  is a monic polynomial, all of whose zeros in  $\mathbb{C}$  have absolute value 1, then prove that every zero is a root of unity.

(2 × 5 = 10 weightage)