D 52325 (Pages : 3) Name..................................

Reg. No..................................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2018

(CUCSS–PG)

Mathematics

MT 3C 12—FUNCTIONAL ANALYSIS – I

(2010 Syllabus Year)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*
Each question carries 1 *weightage.*

1. Prove *or* disprove :

   A sequence (x_n) in the metric space l^2 converges to x in l^2 iff $x_n(s) \to x(s)$ in K for each $j = 1, 2, 3, \ldots$.
2. State Baire's theorem.
3. Define m^{th} Fejer Kernel K_m and show that $\int_{-\pi}^{\pi} K_m(t)\, dt = 2\pi$.
4. Let X be a normed space. Show that if E_1 is open in X and $E_2 \subset X$, then $E_1 + E_2$ is open in X.
5. Show that the closed unit ball in l^1 is convex, but not compact.
6. Show that if X is a finite dimensional normed space, then every linear map from X to itself is continuous.
7. Show that inner product on an inner product space is continuous.
8. Let E be an orthogonal subset of an inner product space X and $0 \notin E$. Show that E is linearly independent.
9. Let $\{x_1, x_2, \ldots\}$ be an orthogonal set in a Hilbert space H. Show that $\sum_{n=1}^{\infty} x_n$ converges in H iff $\sum_{n=1}^{\infty} \| x_n \|^2 < \infty$.
10. Let F be a subspace of an inner product space X and $x \in X$. Show that if $y \in F$ is a best approximation from F to x, then $(x - y) \perp F$.

Turn over

11. Let X be a normed space over K and let $0 \neq a \in X$. Show that there is some $f \in X'$ such that $f(a) = \| a \|$ and $\| f \| = 1$.

12. Show that as a subspace of l^{∞}, c_{00} is not closed.

13. Define Schander basis for a normed space and give an example of a normed space which has a Schander basis.

14. Let X and Y be normed spaces and $F : X \to Y$ be linear. Show that F is continuous iff $g \circ F$ is continuous for every $g \in Y'$.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question carries 2 weightage.

15. Consider $x_0(t) = 1, x_1(t) = t, x_2(t) = t^2$ for $t \in [a, b]$, for $n = 1, 2, \ldots$, let $P_n : C([a, b]) \to C([a, b])$ be a positive linear map. Show that if $P_n(x_j) \to x_j$ uniformly on $[a, b]$ for $j = 0, 1, 2, \ldots$, then $P_n(x) \to x$ uniformly on $[a, b]$ for every x in $C([a, b])$.

16. State and prove Riemann-Lebesgue lemma.

17. Let X be a finite dimensional normed space. Show that every closed and bounded subset of X is compact.

18. Show that the linear functional f on c defined by $f(x) = \lim_{j \to \infty} x(j)$ for $x \in c$ is continuous and $\| f \| = 1$.

19. State and prove Schwarz inequality for an inner product space.

20. Let H be the Hilbert space $L^2([0, 1])$. Show that $\{1, \sqrt{2} \cos \pi t ; \sqrt{2} \cos 2\pi t, \ldots\}$ is an orthonormal basis for H.

21. Let $X = C([-1, 1]), x(t) = 1 - t^2, x_0(t) = 1$ and $x_1(t) = \cos \pi t$, for $t \in [0, 1]$. Show that the best approximation from span $\{x_0, x_1\}$ is $2/3 + \frac{4x_1}{\pi^2}$ to x.

22. Let $X = k^2$ with the norm $\| \ \|_\infty$. Consider $Y = \{(x(1), x(2)) \varepsilon X : x(2) = 0\}$ and define $g \varepsilon Y'$ by $g(x(1), x(2)) = x(1)$. Show that the only Hahn–Banach extension of g to X is $f(x(1), x(2)) = x(1)$.

23. Show that the dual X' of every normed space X is a Banach space.

24. Show that a Banach space cannot have a denumerable basis.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries 4 *weightage.*

25. Show that for $1 \leq p \leq \infty$, the metric space l^p is complete.

26. Give an example of an uncountable orthonormal basis for a Hilbert space.

27. Let X be a normed space. Show that for every subspace Y of X and every $g \varepsilon Y'$, there is a unique Hahn-Banach extension of g to X iff X' is strictly convex.

28. State and prove uniform boundedness principle.

(2 × 4 = 8 weightage)