

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2018

(CUCSS—PG)

Mathematics

MT 1C 02—LINEAR ALGEBRA

(2016 Syllabus Year)

Time : Three Hours

Maximum : 36 Weightage

Part A*Answer all the questions.**Each question carries weightage 1.*

1. Let v be a vector space over a field F . Show that if c is a scalar and o is the zero-vector then $c \cdot o = 0$.
2. Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define :

$$(x, y) + (x_1, y_1) = (x + x_1, 0)$$

$$c(x, y) = (cx, 0)$$

Is V with these operation a vector space ?

3. Let V be the vector space of all $n \times n$ matrices over the field real numbers. Is that the set of all non-invertible matrices A form a subspace of V ?
4. Verify that the Transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by $T(x, y) = ((x - 1) + (y - 1), 2x + 3y)$ is linear transformation or not.
5. Let $\mathcal{B} \{(1, 0, -1), (1, 1, 1), (2, 2, 0)\}$ be a basis for C^3 . Find the dual basis of \mathcal{B} .
6. Find the linear transformation on \mathbb{R}^3 which transform the vector $(1, 5, 9)$ to $(-5, 1, -9)$.
7. Show that similar matrices have the same characteristic polynomial.
8. Prove that every matrix A such that $A^2 = A$ is similar to diagonal matrix.
9. True or False : If a diagonalizable matrix has its only characteristic values zero and one, then it is a projection. Justify.

Turn over

10. Let T be the linear operator on \mathbb{R}^2 , the matrix of which in the standard ordered basis is $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

Let W_1 be the subspace of \mathbb{R}^2 spanned by $(1, 0)$. Show that W_1 is invariant under T .

11. If N is a nilpotent operator on an n -dimensional vector space V , then characteristic polynomial for N is x^n .
12. Let $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ in \mathbb{R}^2 . Check whether \mathbb{R}^2 with the product $\langle \alpha | \beta \rangle = x_1 y_1 - x_2 y_1 - x_1 y_2 + 4x_2 y_2$ is an inner product space or not.
13. Show that if S is any subset of a vector space V , then its orthogonal complement S^\perp is a subspace of V .
14. If $V(F)$ be a vector space of polynomials in t with the product is defined by $\langle p(t), q(t) \rangle = \int_0^1 p(t) q(t) dt$. If $p(t) = t + 2$ then find $\|p\|^2$.

(14 × 1 = 14 weightage)

Part B

*Answer any seven questions.
Each question carries weightage 2.*

15. Find the co-ordinates of the vector $(2, 1, -6)$ of \mathbb{R}^3 relative to the basis $\alpha_1 = (1, 1, 2)$, $\alpha_2 = (3, -1, 0)$, $\alpha_3 = (2, 0, -1)$.
16. Let V be the space of 2×2 matrices over F . Find a basis $\{A_1, A_2, A_3, A_4\}$ for V such that $A_j^2 = A_j$ for each j .
17. Let W_1 and W_2 be the subspace of a finite dimensional vector space V . Prove that $(W_1 \cap W_2)^\circ = W_1^\circ + W_2^\circ$.
18. If S is any subset of a finite dimensional vector space V , then prove that $(S^0)^\circ$ is the subspace spanned by S .

19. Let T be the linear operator on \mathbb{R}^4 which is represented in the standard order matrix $\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix}$.

Under what condition on a , b and c is T diagonalizable?

20. Let T be a diagonalizable linear operator on finite dimensional vector space V , and let W be a subspace which is invariant under T . Prove that the restriction operator T_W is diagonalizable.
21. If $W_1 = \{(x, 0, z) : x, z \in \mathbb{R}\}$, $W_2 = \{(0, y, z) : y, z \in \mathbb{R}\}$ be two subspaces of \mathbb{R}^3 , then prove that $\mathbb{R}^3 = W_1 + W_2$ but not $\mathbb{R}^3 = W_1 \oplus W_2$.
22. Let E be the projection and T be a linear operator on a vector space V . Show that the range and null space of E are invariant under T if and only if $ET = TE$.
23. Let S be a orthogonal set of non-zero vectors in an inner product space V . Show that S is a linearly independent set.
24. State and prove Bessel's inequality.

(7 × 2 = 14 weightage)

Part C

*Answer any two questions.
Each question carries 4 weightage.*

25. Let V be the vector space of all 2×2 matrices over the field. Let W_1 be the set of matrices of the form $\begin{pmatrix} x & -x \\ y & z \end{pmatrix}$ and let W_2 be that set of matrices of the form $\begin{pmatrix} a & b \\ -a & c \end{pmatrix}$:
- (a) Prove that W_1 and W_2 are subspace of V .
- (b) Find the dimensions of W_1 , W_2 , $W_1 + W_2$ and $W_1 \cap W_2$.
26. Let W be the subspace of \mathbb{R}^5 spanned by the vectors $\alpha_1 = (2, -2, 3, 4, -1)$, $\alpha_2 = (-1, 1, 2, 5, 2)$, $\alpha_3 = (0, 0, -1, -1, 3)$, $\alpha = (1, -1, 2, 3, 0)$. Describe W^0 , the annihilator of W .

Turn over

27. Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V . Then T is diagonalizable if and only if the minimal polynomial for T has the form $p(x) = (x - c_1)(x - c_2) \dots (x - c_k)$, where c_1, c_2, \dots, c_k are distinct elements of F .
28. Apply the Gram-Schmidt process to the vector process to the vector $\beta_1 = (3, 0, 4)$, $\beta_2 = (-1, 0, 7)$, $\beta_3 = (2, 9, 11)$, to obtain an orthonormal basis for \mathbb{R}^3 with the standard inner product.

(2 × 4 = 8 weightage)