**C 32338** (Pages : 3) Name..................................

Reg. No..............................

## FIRST SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2017

(CUCSS)

Mathematics

MT 1C 04—NUMBER THEORY

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

### Part A

*Answer* **all** *questions.*

*Each question carries a weightage of 1.*

1. Find all integers $n$ such that $\phi(n)=\frac{n}{2}$.

2. Give an example of an arithmetical function which is multiplicative but not completely multiplicative.

3. Prove that, for any arithmetical functions $\alpha$ and $\beta'$,

$$\alpha \circ (\beta \circ F) = (\alpha * \beta) \circ F.$$

Where F denotes a real or complex-valued function defined on the positive real axis $(0, \infty)$ such that $F(x)=0$ for $0<x<1$.

4. If $x \geq 1$, prove that

$$\sum_{n \leq x} \frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right).$$

5. Prove that $[x]+\left[x+\frac{1}{2}\right]=[2x]$.

6. Define the Chebyshev's functions $\Psi(x)$ and $\vartheta(x)$

7. For $x \geq 2$, prove that $\vartheta(x)=\pi(x)\log x-\int_2^x \frac{\pi(t)}{t}\,dt.$

Turn over

8. If $a > 0$ and $b > 0$, prove that $\pi(ax)/\ \pi(bx) \sim a/b$ as $x \to \infty$.

9. For $x \geq 1$, prove that

$$\sum_{n \leq x} \frac{\wedge(n)}{n} = \log x + O(1).$$

10. Define the Legendre's symbol $(n/p)$. Prove that $(n/p)$ is a periodic function of $n$ with period $p$.

11. Evaluate (73/383).

12. If P and Q are odd positive integers, prove that $(m/p)(n/p) = (mn/p)$.

13. What is meant by a cryptosystem.

14. What are the disadvantages of deterministic encryption

(14 × 1 = 14 weightage)

## Part B

*Answer any* **seven** *questions.*
*Each carries a weightage of* 2.

15. If $n \geq 1$, prove that

$$\log n = \sum_{d/n} \wedge(d).$$

16. State and prove the Selberg identify.

17. For all $n \geq 2$, prove that

$$\left| \sum_{n \leq x} \frac{\mu(n)}{n} \right| < 1.$$

18. State and prove Euler's summation formula.

19. For $x > 0$, prove that

$$0 \leq \frac{\psi(x)}{x} - \frac{\vartheta(x)}{x} \leq \frac{(\log x)^2}{2\sqrt{x}\,\log 2}.$$

20. If $\{a(n)\}$ is a non-negative sequence such that

$$\sum_{n \le x} a(n)\left[\frac{x}{n}\right] = x \log x + O(x) \text{ for all } x \ge 1,$$

then prove that there is a constant $B > 0$ such that $\sum_{n \le x} a(n) \le Bx$ for all $x \ge 1$.

21. If $A(x) = \sum_{n \le x} \frac{\mu(n)}{n}$, prove that the relation $A(x) = O(1)$ as $x \to \infty$ implies the prime number theorem.

22. Let $p$ be an odd prime. Prove that $\sum_{r=1}^{p-1} r^2 (r/p) = p \sum_{r=1}^{p-1} r(r/p)$ if $p \equiv 3 \pmod 4$.

23. In the 27-letter alphabet (with blank = 26), using the affine enciphering transformation with key $a = 13$, $b = 9$, encipher the message "HELP ME".

24. Explain briefly about RSA cryptosystem.

(7 × 2 = 14 weightage)

## Part C

*Answer any* **two** *questions.*
*Each carries a weightage of* 4.

25. Show that the set of all arithmetical functions $f$ with $f(1) \neq 0$ forms an Abelian group with respect to the Dirichlet product.

26. For every integer $n \ge 2$, prove that

$$\frac{1}{6} \frac{n}{\log n} < \pi(n) < 6 \frac{n}{\log n}.$$

27. State and prove Gauss' Lemma.

28. The message "KVW ? TA! KJB ? FVR". (The blanks after ? and R are part of the message, but the final · is not.) It is known that a linear enciphering transformation is used with a 30-letter alphabet, in which A-Z have numerical equivalents 0 – 25, blank = 26, ? = 27, ! = 28, · = 29. It is also known that the first five letters of the plain text are "C.I.A". Find the deciphering matrix and the full plain text message.

(2 × 4 = 8 weightage)