C 80715 (Pages : 3) Name.................................

Reg. No.................................

# FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, MARCH 2020

(CUCSS)

Mathematics

## MT4 E14—DIFFERENTIAL GEOMETRY

Time : Three Hours Maximum : 36 Weightage

### Part A

*Answer* **all** *questions.*
*Each question carries weightage* 1.

1. Describe the level set at $c = 2$ for the function $f(x_1, x_2) = x_1 + x_2$.

2. Sketch the vector field on $\mathbb{R}^2$ given by $X(p) = (1, 0)$.

3. Show that the plane curve $x_1 + 2x_2 + 3x_3$ is a level set of some function $f$.

4. Give an orientation on the 2–sphere.

5. Define Gauss map.

6. Give an example of a vector field along a parametrized curve.

7. Let $\alpha$ be a geodesic on S. Show that $\dot{\alpha}' = 0$ along $\alpha$.

8. Define parallel vector field along a parametrized curve $\alpha$.

9. Show that $\nabla_{v+w} f = \nabla_v f + \nabla_w f$ for all smooth functions $f$.

10. Find $L_p(v)$ for $p = (1, 0, 0)$ and $v = (2, 1, 0)$ on the sphere $x_1^2 + x_2^2 + x_3^2 = 1$.

11. Find the length of the parametrized curve $\alpha : [0, \pi] \to \mathbb{R}^2$ given by $\alpha(t) = (\cos t, \sin t)$.

12. Find the Gaussian curvature of an $n$–surface S at $p$ where the principal curvatures are $1, 1/2, 1/3$..

Turn over

13. Give an example of a parametrized 1–surface.

14. Describe the stereographic projection from a sphere to a plane.

(14 × 1 = 14 weightage)

## Part B

*Answer any* **seven** *questions.*
*Each question carries weightage* 2.

15. Sketch the gradient field of the function $f(x_1, x_2) = x_1^2 + x_2^2$.

16. Describe the tangent space of the unit circle at $p = (1, 0)$.

17. Find the spherical image of the 2–sphere $x_1^2 + x_2^2 + x_3^2 = 2$.

18. With the usual notations prove that $(f\dot{X}) = f'X + f\dot{X}$

19. Let S be an $n$–surface. Prove that if S contains a line segment $\alpha(t) = p + tv$ for $t \in [0, 1]$ then $\alpha$ is a geodesic on S.

20. Let $X'$ denote the covariant derivative of a vector field X along a parametrized Curve $\alpha$. Show that $(X + Y)' = X' + Y'$

21. Find the curvature $k$ of the plane curve $\frac{x_1^2}{3} + \frac{x_2^2}{5} = 1$.

22. Let $k(p)$ be the curvature of a palne curve at $p$ and $v$ be a non-zero vector tangent to the plane curve at $p$. Show that $k(p) = L_p(v) \cdot v / \| v^2 \|$.

23. Let N $(v)$ be the normal section of an $n$—surface determined by a unit vector $v$. Show that N $(v)$ is isomorphic to a copy of $\mathbb{R}^2$.

24. Described the parametrized torus in $\mathbb{R}^4$.

(7 × 2 = 14 weightage)

## Part C

*Answer any* **two** *questions.*
*Each question carries weightage 4.*

25. (a) Define integral curve of a vector field.

(b) Show that for the vector field $X(x_1, x_2) = (-x_2, x_1)$, the parametrized curve a $\alpha(t) = (\cos t - \sin t, \cos t + \sin t)$ in an integral curve on it.

(c) Find an integral curve for the vector field given by $X(x_1, x_2) = (x_2, x_1)$.

26. Let S be an $n$–surface in $\mathbb{R}^{n+1}$ and let X be a smooth tangent vector field on S and let $p \in S$. Show that there exists an open interval I containing 0 and a parametrized curve $\alpha : I \to S$ such that :

(a) $\alpha(0) = p$ ; and

(b) $\dot{\alpha}(t) = X(\alpha(t))$ for $t \in I$.

27. Let S be an $n$–surface in $\mathbb{R}^{n+1}$ and $\alpha : I \to S$ be a prarametrized curve in S. Let X be a vector field along $\alpha$. Prove that :

(a) X is parallel along $\alpha$ if and only if X satisfies the differential equation

$$\dot{X}(t) + \left(X(t) \cdot \dot{N}(\alpha(t))\right). N(\alpha(t)) = 0 \text{ for all } t.$$

(b) If X is a solution of the above differential equation then X is tangent to S along $\alpha$.

28. (a) Define second fundamental form of an oriented $n$–surface.

(b) Let S be a compact $n$–surface in $\mathbb{R}^{n+1}$. Snow that there exists $p \in S$ such that the second fundamental form at $p$ is definite.

(2 × 4 = 8 weightage)