C 4749 (Pages : 3) Name.................................

Reg. No..............................

SECOND SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2021

(CBCSS)

Mathematics

MT 2C 07—REAL ANALYSIS—II

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *questions.*
Each question carries a weightage of 1.

1. Let A be the set of irrational numbers in the interval [0, 1], Prove that $m^*(A) = 1$.
2. Prove that a monotone function that is defined on an interval is a measurable function.
3. If $\{f_k\}_{k=1}^{n}$ is a finite family of measurable functions with common domain E, then prove that the functions max $\{f_1 ..., f_n\}$ and min $\{f_1, ..., f_n\}$ are also measurable.
4. Let f and g be bounded measurable functions on a set of finite measure E. If $f \leq g$ on E, then prove that $\int_E f \leq \int_E g$.
5. Let E be a set of finite measure and let $\delta > 0$ be given. Prove that E is the disjoint union of a finite collection of sets, each of which has measure less than δ.
6. If $\{f_n\} \to f$ in measure on E, then prove that there is a subsequence $\{f_{n_k}\}$ that converges pointwise a.e. on E to f.

Turn over

7. Find the upper and lower derivatives of f at $x = 0$ for the function $f(x) = |x|$, for all real numbers x.

8. Give an example of a Cauchy sequence of real numbers that is not rapidly Cauchy.

(8 × 1 = 8 weightage)

Part B

*Answer any **six** questions by choosing **two** questions from each unit.*
Each question carries a weightage of 2.

Unit I

9. Show that every interval is a Borel set.

10. Show that the Cantor set is an uncountable set of measure zero.

11. Let $\{f_n\}$ be a sequence of measurable functions on E that converges point wise a.e. on E to the function f. Prove that f is measurable.

Unit II

12. Show that the function f defined on [0, 1] by $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational is not Riemann integrable over [0, 1], but it is Lebesgue integrable over [0, 1].

13. State and prove the Monotone Convergence theorem.

14. Let E have finite measure, $\{f_n\} \to f$ in measure on E and g is a measurable function on E that is finite a.e. on E. Prove that $\{f_n \cdot g\} \to f.g$ in measure.

Unit III

15. Let f and g be real-valued functions on (a, b). Show that, on (a, b),

$$\underline{D}f + \underline{D}g \leq \underline{D}(f+g) \leq \overline{D}(f+g) \leq \overline{D}f + \overline{D}g.$$

16. State and prove Jensen's Inequality.

17. Let E be a measurable set and $1 \leq p \leq \infty$. If the functions f and g belong to $L^P(E)$, then prove that their sum $f+g$ also belong to $L^P(E)$. Also prove that

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. Define the outer measure $m^*(A)$ of a set $A \subset \mathbb{R}$ and give an example. Prove that the outer measure of an interval is its length.

19. (a) State and prove Egoroff's theorem.

    (b) Let f be a bounded function on a set of finite measure E. Prove that f is Lebesgue integrable over E if and only if it is measurable.

20. (a) Let E have measure zero. Show that if f is a bounded function on E, then f is measurable and $\int_E f = 0$.

    (b) Let E be of finite measure. Suppose the sequence of functions $\{f_n\}$ is uniformly integrable over E. If $\{f_n\} \to f$ pointwise a.e. on E, then prove that f is integrable over E and $\lim_{n \to \infty} \int_E f_n = \int_E f$.

21. (a) If the function f is monotone on the open interval (a, b), then prove that it is differentiable almost everywhere on (a, b).

    (b) Prove that a function f defined on a closed, bounded interval $[a, b]$ is absolutely continuous on $[a, b]$ if and only if it is an indefinite integral over $[a, b]$.

(2 × 5 = 10 weightage)