C 32337 (Pages : 3) Name..................................

Reg. No..............................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2017

(CUCSS)

Mathematics

MT 1C 03—REAL ANALYSIS—I

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A (Short Answer Questions)

Answer **all** *questions.*

Each question has weightage 1.

1. Prove that the neighbourhood of a point p, in a metric space X, is an open set.

2. Shoe that the set of limit points of a bounded set is bounded.

3. Let $E = \left(p \in \mathbb{Q} : 2 < p^2 < 3\right)$. Show that E is closed and bounded in $\mathbb{Q}$. Is E compact ? Justify your answer.

4. Let X be a compact metric space and $f : X \to \mathbb{R}^k$. Is $f(X)$ closed ? Justify your answer.

5. Identify the type of discontinuity at $x = 0$ of the function f defined by $f(x) = \begin{cases} -x^2 \text{ if } x \le 0 \\ 5x - 4 \text{ if } 0 < x \le 1 \\ 4x^2 - 3x \text{ if } 1 < x < 2 \end{cases}$.

6. Let f be defined on $[a, b]$ by $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$ in $[a, b]$. Is f uniformly continuous on $[a, b]$? Justify your answer.

7. If f is differentiable in (a, b) and $f'(x) \le 0$ for all $x \in (a, b)$, prove that f is monotonically decreasing.

8. Prove that $\lim_{x \to 0} \dfrac{\cos x - 1}{x} = 0$.

9. Verify whether the Mean Value Theorem is applicable to $f(x) = 2x^2 + 3x + 4$ in [1, 2].

Turn over

10. If f is monotonic on $[a, b]$ and if α is continuous on $[a, b]$ then prove that $f \in (R)(\alpha)$ on $[a, b]$.

11. If f is continuous on $[a, b]$ and α is monotonic increasing on $[a, b]$, prove that there exists $c \in [a, b]$ such that $\int_a^b f\, d\alpha = f(c)[\alpha(b) - \alpha(a)]$.

12. Let γ be the curve in the complex plane defined by $\gamma(t) = e^{2it}$. Verify whether γ is rectifiable.

13. If $\{f_n\}$ is a sequence of functions differentiable on $[a, b]$ such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$ and if $\{f_n'\}$ converges uniformly on $[a, b]$, prove that $\{f_n\}$ converges uniformly on $[a, b]$.

14. Let $\mathscr{F}$ denote the family of equicontinuous functions defined on a set E in a metric space and $f_\lambda \in \mathscr{F}$. Is f_λ uniformly continuous ? Justify your answer.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions from the following* **ten** *questions.*

Each question has weightage 2.

15. Let X be a metric space and $E \subset X$. Prove that $\bar{E}$ is the smallest closed subset of X that contains E.

16. If $K \subset Y \subset X$ and X is a metric space, prove that K is compact relative to X if and only if K is compact relative to Y.

17. Let F be a closed subset of the metric space X. If K is a compact subset of X, prove that $F \cap K$ is compact.

18. Let $X = [0, 2\pi)$ and Y is the unit circle centered at the origin. Let $f : X \to Y$ be defined by $f(t)(\cos t, \sin t)$. If f continuous ? Does f^{-1} exist ? If it exists, is it continuous ? Justify your answer.

19. If f is continuous on $[a, b]$, prove that f is uniformly continuous on $[a, b]$.

20. State and prove that chain rule for differentiate.

21. If $f \in \mathcal{R}(\alpha)$ on $[a, b]$, $m \leq f \leq M$, ϕ is continuous on $[m, M]$ and $h(x) = \phi(f(x))$ on $[a, b]$, prove that $h \in \mathcal{R}(\alpha)$ on $[a, b]$.

22. State and prove the fundamental theorem of calculus.

23. If $\{f_n\}$ is sequence of continuous functions defined on E in a metric space and if $f_n \to f$ uniformly on E prove that f is continuous on E.

24. Let $f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2}$, where $0 \leq x \leq 1, n = 1, 2, ...$ Is $\{f_n\}$ equicontinuous? Justify your claim.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions from the following* **four** *questions.*
Each question has weightage 4.

25. (a) Let $f : X \to Y$ be continuous where X and Y are metric spaces. If E is a connected subset of X prove that $f(E)$ is connected.

    (b) Prove that $\mathbb{R}^k$ contains a countable dense subset.

26. Suppose α increases monotonically, $\alpha' \in \mathcal{R}$ on $[a, b]$ and f is a bounded real function on $[a, b]$. Show that $f \in \mathcal{R}(\alpha)$ on $[a, b]$ if and only if $f\alpha' \in \mathcal{R}$.

27. (a) Define equicontinuous family of functions.

    (b) If K is a compact metric space, $f_n \in \mathcal{C}(K)$, for $n = 1, 2, ...$ and $\{f_n\}$ converges uniformly on K then prove that $\{f_n\}$ is equicontinuous on K.

28. State and prove the Stone-Weierstrass theorem.

(2 × 4 = 8 weightage)