D 72973 (Pages : 3) Name....................................

Reg. No....................................

FIRST SEMESTER M.A./M.Sc./M.Com. DEGREE EXAMINATION DECEMBER 2019

(CBCSS)

Mathematics

MTH 1C 05—NUMBER THEORY

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*

Each question carries a weightage of 1.

1. If $n \geq 1$, prove that

$$\sum_{d/n} \mu(d) = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ if } n > 1 \end{cases}.$$

2. Prove that the equation

$$f(n) = \sum_{d/n} g(d).$$

implies $g(n) = \sum_{d/n} f(d)\, \mu\left(\frac{n}{d}\right)$.

3. For $x \geq 1$, prove that $\sum_{n \leq x} \mu(n) \left[\frac{x}{n}\right] = 1.$

4. Define the Chebyshev's functions $\psi(x)$ and $\vartheta(x)$.

5. Prove that $\lim_{x \to \infty} \left(\frac{M(x)}{x} - \frac{H(x)}{x \log x}\right) = 0.$

6. Determine whether -104 is a quadratic residue or non-residue of the prime 997.

Turn over

7. What is meant by enciphering transformation and deciphering transformation in a cryptosystem?

8. How can a message be authenticated in electronic communication.

(8 × 1 = 8 weightage)

Part B

Answer **six** *by choosing two questions from each unit.*
Each question carries a weightage of 2.

Unit I

9. Define a completely multiplicative function. Give an example of a multiplicative function which is not completely multiplicative.

10. For the Euler totient function ϕ, prove that $\phi^{-1}(n) = \prod_{p/n} (1-p)$.

11. If $x \geq 1$ prove that

$$\sum_{n \leq x} \frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right),$$

where C is the Euler's constant.

Unit II

12. For $x \geq 2$, prove that $\vartheta(x) = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt$.

13. For $n \geq 1$, prove that the n^{th} prime p_n satisfy the inequality

$$p_n < 12\left(n \log n + n \log \frac{12}{e}\right).$$

14. If $A(x) = \sum_{n \leq x} \frac{\mu(n)}{n}$, then prove that the relation $A(x) = 0(1)$ as $x \to \infty$. implies the prime number theorem.

Unit III

15. Let p be an odd prime. Prove that for all integers n, $(n/p) \equiv n^{(p-1)/2} \pmod{p}$.

16. Let p be an odd prime. Prove that $\sum_{r=1}^{p-1} r\,(r/p) = 0$ if $p \equiv 1 \pmod{4}$.

17. Find the inverse of the matrix $\begin{pmatrix} 1 & 3 \\ 4 & 3 \end{pmatrix} \bmod 5$.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. (a) If f and g are multiplicative, prove that their Dirichlet product $f * g$ is also multiplicative.

    (b) Let f be multiplicative. Prove that f is completely multiplicative if and only if

    $f^{-1}(n) = \mu(n) f(n)$ for all $n \geq 1$.

19. Let p_n denote the n^{th} prime. Prove that the following asymptotic relations are logically equivalent :

    (i) $\lim_{x \to \infty} \dfrac{\pi(x) \log x}{x} = 1.$

    (ii) $\lim_{x \to \infty} \dfrac{\pi(x) \log \pi(x)}{x} = 1.$

    (iii) $\lim_{n \to \infty} \dfrac{p_n}{n \log n} = 1.$

20. State and prove Gauss's Lemma.

21. Explain briefly about public key cryptosystem. Discuss the advantages and disadvantages of public key cryptosystem over classical cryptosystem.

(2 × 5 = 10 weightage)