C 2049 (Pages : 3) Name....................................

Reg. No..................................

FOURTH SEMESTER M.Sc. DEGREE (REGULAR) EXAMINATION MARCH 2021

(CBCSS)

Mathematics

MTH 4E 09—DIFFERENTIAL GEOMETRY

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *questions.*
Each question has weightage 1.

1. Find and sketch the gradient field of the function $f(x_1, x_2) = x_1 + x_2$.

2. Show that if $\alpha : I \to \mathbb{R}^{n+1}$ is a parametrized curve with constant speed then $\ddot{\alpha}(t) \perp \dot{\alpha}(t), \forall t \in I$.

3. Define Euclidean parallel and Levi-Civita parallel vector fields.

4. Define (i) Global parametrization ; (ii) Circle of curvature ; and (iii) Radius of curvature of a plain curve.

5. Find the length of the parametrized curve $\alpha : I \to \mathbb{R}^3$ where $\alpha(t) = (\cos 3t, \sin 3t, 4t), I = [-1, 1]$.

6. Compute $\nabla_{\overline{v}} f$ where $f : \mathbb{R}^2 \to \mathbb{R}$ and $\overline{v} \in \mathbb{R}^2_p, p \in \mathbb{R}^2$ given by $f(x_1, x_2) = {x_1}^2 - {x_2}^2$, $\overline{v} = (1, 1, \cos\theta, \sin\theta)$.

Turn over

7. Define a parametrized n-surface. Write the map which represent the parameterized torus in $\mathbb{R}^4$.

8. State inverse function theorem for n-surface.

(8 × 1 = 8 weightage)

Part B

Answer **two** *questions from each unit in this part.*
Each question has weightage 2.

Unit 1

9. Define (i) Level set ; and (ii) Graph of a function. Sketch the Graph f and level set of the function $f(x_1, x_2) = x_1^2 - x_2^2$.

10. Find the integral curve through $P(1, 0)$ and $P(a, b)$ for the vector field

$\bar{X}(p) = \left(x_1, x_2, -2x_2, \tfrac{1}{2}x_1\right)$ on $U \subseteq \mathbb{R}^2$.

11. Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f and $c = f(p)$. Then show that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $\left[\nabla f(p)\right]^{\perp}$.

Unit 2

12. Prove that the Weingarten map L_p is self-adjoint.

13. Let $\bar{X}, \bar{Y}$ be two smooth tangent vector fields on S and $f : U \to \mathbb{R}$ any smooth function and $\alpha : I \to S$ is a parametrized curve with $\alpha(t_0) = p$ and $\dot{\alpha}(t_0) = \bar{v}$. Then prove that :

(i) $D_v - \left(\bar{X} + \bar{Y}\right) = D_v - \bar{X} + D_v - \bar{Y}$; and (ii) $\nabla_v - \left(\bar{X} \cdot \bar{Y}\right) = D_v - \bar{X} \cdot \bar{Y}(p) + \bar{X}(p) \cdot D_v - \bar{Y}$.

14. Find the global parametrization and curvature K of the circle $(x_1 - a)^2 + (x_1 - b)^2 = r^2$, oriented by the out ward normal $\nabla f / \| \nabla f \|$.

Unit 3

15. Find the Gaussian curvature for the ellipsoid $\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}=1$; $a, b, c \neq 0$ oriented by its outward normal.

16. Prove that on each compact oriented n-surface S in $\mathbb{R}^{n+1}$, there exists a point p such the second fundamental form of p is definite.

17. Show that $\varphi : \mathrm{U} \to \mathbb{R}^3$ where $\mathrm{U} = \{(\theta, \phi) \in \mathbb{R}^2 : 0 < \phi < \pi\}$ and $a > b > 0$ given by

$\varphi(\theta, \phi) = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi)$ is a parametrized 2-surface in $\mathbb{R}^3$.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question has weightage 5.

18. Let S be a compact connected oriented n-surface in $\mathbb{R}^{n+1}$ exhibited as a level set $f^{-1}(c)$ of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\nabla f(p) \neq 0$ for all $p \in \mathrm{S}$. Then the Gauss map maps S onto the unit sphere S^n.

19. Let C be an a connected oriented plane curve and let $\beta : \mathrm{I} \to \mathrm{C}$ be a unit speed global parameterization of C. Then β is either one to one or periodic. Also show that β is periodic if and only if C is compact.

20. (a) Prove that if V is a finite dimensional vector space with dot product and $\mathrm{L} : \mathrm{V} \to \mathrm{V}$ a self-adjoint linear transformation on V. Then there exists an orthonormal basis for V consisting of eigenvectors of L.

    (b) Find the Gaussian curvature of a cylinder over a plain curve.

21. Find the Gaussian curvature of the parametrized 2-surface

$\varphi(\theta, \phi) = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi)$ in R^3.

(2 × 5 = 10 weightage)