C 80706 (Pages : 3) Name..................................

Reg. No..................................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, MARCH 2020

(CUCSS)

Mathematics

MT4 E05—MEASURE AND INTEGRATION

Time : Three Hours Maximum : 36 Weightage

Part A

Short Answer Questions.
Answer **all** *questions (1-14).*
Each question has weightage 1.

1. Let X be an uncountable set and let

   $\mathcal{M} = \{E \subseteq X : \text{either E of } E^c \text{ is at most countable}\}$.

   Prove that $\mathcal{M}$ is a σ-algebra in X.

2. Let X be a topological space with indiscrete topology. Find all Borel sets of X.

3. Let f be a real valued function on a measurable space X. If $\{x : f(x) \geq r\}$ is measurable for every rational number r, then prove that f is measurable.

4. Let $(X, \mathcal{M}, \mu)$ be a measurable space and let s be a non-negative measurable function on X. For $E \in \mathcal{M}$, define

   $\varphi(E) = \int_E s \, d\mu.$

   Prove that φ is a measure on $\mathcal{M}$.

5. Define total variation of a measure μ. If μ is a positive measure, then prove that $|\mu| = \mu$.

6. Let λ_1, λ_2 be measures on a σ-algebra $\mathcal{M}$. If $\lambda_1 \perp \lambda_2$, then prove that $|\lambda_1| \perp |\lambda_2|$.

7. Let $\lambda_1, \lambda_2, \mu$ be measures on a σ-algebra $\mathcal{M}$ and let μ be positive. If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then prove that $\lambda_1 \perp \lambda_2$.

8. Find the Lebesgue measure of the rectangle $\{(x, y) : 1 \leq x \leq 3, 5 \leq y \leq 10\}$.

9. Prove that the positive variation of a real measure μ is bounded.

10. Define Jordan decomposition of a real measure.

Turn over

11. Let $f \in L^1(\mathbb{R}^k)$ and f be continuous at $x \in \mathbb{R}^k$. Prove that x is a Lebesgue point of f.

12. Define nicely shrinking sets and give an example of it.

13. Give an example of a differentiable function f such that $f' \notin L^1$.

14. Let (X, T) and (Y, S) be measure spaces where $X = \{a,b,c\}, T = \{X, \phi, \{b\}, \{a,c\}\}$, $Y = \{e,f\}$ and $S = \{Y, \phi\}$. Find $T \times S$.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *from the following ten questions (15-24).*
Each question has weightage 2.

15. Let f be a complex measurable function on a measurable space X. Prove that there is a complex measurable function α on X such that $|\alpha| = 1$ and $f = \alpha|f|$.

16. Let X be a measurable space and let $f : X \to [0,\infty]$ be a measurable function. Prove that there exists a simple measurable functions s_n on X such that

 (i) $0 \le s_1 \le s_2 \le ..... \le f$,

 (ii) $s_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$.

17. Let X be a measurable space and let $f_n : X \to [-\infty,\infty]$ be measurable functions for $n = 1,2,....m$. If $g = \sup\{f_1, f_2, ...f_m\}$, then prove that g is measurable.

18. Let $(X, \mathcal{M}, \mu)$ be a measurable space, $f : X \to [0,\infty]$ is measurable and $E \in \mathcal{M}$. If $\int_E f \, d\mu = 0$, then prove that $f = 0$ a.e. in E.

19. Show that every compact subset of $\mathbb{R}^1$ is the support of a Borel measure.

20. Let μ be a real measure on a σ-algebra. Prove that $\mu^+ \perp \mu^-$.

21. Prove that Lebesgue decomposition of a complex measure is unique.

22. Let $f \in L^1(\mathbb{R}^1)$ and let $F(x) = \int_{-\infty}^{x} f \, dm$. Prove that $F'(x) = f(x)$ at every Lebesgue point of f.

23. Let (X, S) and (Y, I) be measurable spaces and let $E \in S \times I$. Prove that $E_x = \{y : (x, y) \in E\} \in I$.

24. Let (X, S) and (Y, I) be measurable spaces and let f be an (S × I)-measurable function on X × Y. Prove that for each $y \in Y, f_y(x) = f(x, y)$ is a S-measurable function.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *from the following four questions (25-28).*
Each question has weightage 4.

25. State and prove Lebesgue monotone convergence theorem.

26. Let (X, μ) be a measurable space, f be a real valued Lebesgue integrable function with respect to μ and $\varepsilon > 0$. Prove that there exist functions u and v on X such that $u \leq f \leq v$, u is upper semicontinuous and bounded above, v is lower semicontinuous and bounded below and $\int_X (v - u)\, d\mu < \varepsilon$.

27. If $f \in L^1(\mathbb{R}^k)$, then prove that almost every $x \in \mathbb{R}^k$ is a Lebesgue point of f.

28. Let (X, S) and (Y, T) be measurable spaces. Prove that S × T is the smallest monotone class which contains all elementary sets.

(2 × 4 = 8 weightage)