

**SIXTH SEMESTER B.Sc. DEGREE EXAMINATION, MARCH 2019**

(CUCBCSS)

Mathematics

MAT 6B 12—NUMBER THEORY AND LINEAR ALGEBRA

Time : Three Hours

Maximum : 120 Marks

**Section A***Answer all the twelve questions.**Each question carries 1 mark.*

1. State the division algorithm.
2. State the Fundamental Theorem of Arithmetic.
3. Give an example to show that  $a^2 \equiv b^2 \pmod{n}$  does not imply  $a \equiv b \pmod{n}$ .
4. Define multiplicative functions.
5. If  $x$  is a real number, what are the possible values of  $[x] + [-x]$ ?
6. Define Euler phi function.
7. Find the highest power of 5 dividing  $1000!$ .
8. Define subspace of a vector space  $V$ .
9. Find Span  $S$  where  $S = \{(1, 0, 0)\} \subseteq \mathbb{R}^3$ .
10. Define a linear transformation.
11. Give two different bases for  $\mathbb{R}^2$ .
12. Define null space of a linear transformation.

(12 × 1 = 12 marks)

**Section B***Answer any ten out of fourteen questions.**Each question carries 4 marks.*

13. Prove that the square of any odd integer is of the form  $8k + 1$  where  $k$  is an integer.
14. Prove that if  $\gcd(a, b) = d$ , then  $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$  where  $a, b$  are integers.

**Turn over**

15. Let  $\gcd(a, b) = 1$ . Prove that  $\gcd(a + b, a - b) = 1$  or  $2$ .
16. Determine all solutions of the Diophantine equation  $56x + 72y = 40$ .
17. Prove that if  $a \equiv b \pmod{n}$ , then  $a + c \equiv b + c \pmod{n}$  and  $ac \equiv bc \pmod{n}$ .
18. Solve the congruence  $18x \equiv 30 \pmod{42}$ .
19. Prove that if  $p$  is a prime, then  $a^p \equiv a \pmod{p}$  for any integer  $n$ .
20. Prove that  $\tau$  is a multiplicative function.
21. Prove that the intersection of two subspaces of a vector space  $V$  is again a subspace of  $V$ .
22. Check whether the vectors  $(1, 1, 0)$  and  $(2, 5, 3)$  and  $(0, 1, 1)$  in  $\mathbb{R}^3$  are linearly independent.
23. Let  $W$  be a subspace of a vector space  $V$ . Prove that  $\dim W = \dim V$  if and only if  $V = W$ .
24. Prove that  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  defined by  $f(a, b) = (a + b, a - b, b)$  is a linear transformation.
25. Let  $V$  and  $W$  be vector spaces. Prove that if the linear mapping  $f: V \rightarrow W$  is injective and  $\{v_1, v_2, \dots, v_n\}$  is a linearly independent subset of  $V$ , then  $\{f(v_1), f(v_2), \dots, f(v_n)\}$  is a linearly independent subset of  $W$ .
26. Let  $V$  and  $W$  be vector spaces. Prove that the linear mapping  $f: V \rightarrow W$  is injective if and only if  $\text{Ker } f = \{0\}$ .

(10 × 4 = 40 marks)

### Section C

Answer any **six** out of nine questions.

Each question carries 7 marks.

27. Prove that  $\sqrt{2}$  is irrational.
28. Prove that the sequence of primes is infinite.
29. Prove that the linear congruence  $ax \equiv b \pmod{n}$  has a solution if and only if  $d \mid b$  where  $d = \gcd(a, n)$ . If  $d \mid b$  prove that the congruence has  $d$  mutually incongruent solutions modulo  $n$ .

30. Use Chinese Remainder Theorem to find the smallest non-negative solution of the given system of

$$\begin{aligned} \text{congruences : } & x \equiv 2 \pmod{3} \\ & x \equiv 3 \pmod{5} \\ & x \equiv 2 \pmod{7}. \end{aligned}$$

31. If  $n$  and  $r$  are positive integers with  $1 \leq r < n$ , then the binomial co-efficient  $\binom{n}{r} = \frac{n!}{r!(n-r)!}$  is an integer.

32. Let  $V$  be a vector space over a field  $F$ . Prove the following :

(a)  $\lambda 0_V = 0_V \forall \lambda \in F$ .

(b)  $0_F x = 0_V \forall x \in V$ .

(c) If  $\lambda x = 0_V$ , then either  $\lambda = 0_F$  or  $x = 0_V$ .

33. Let  $S$  and  $T$  be two non-empty finite subsets of a vector space  $V$  such that  $S \subseteq T$ . Prove the following :

(a) If  $T$  is linearly independent, then so is  $S$ . (b) If  $S$  is linearly dependent, then so is  $T$ .

34. Let  $V$  be a finite dimensional vector space. If  $G$  is a finite spanning set of  $V$  and if  $I$  is a linearly independent subset of  $V$  such that  $I \subseteq G$ , prove that there is a basis  $B$  of  $V$  such that  $I \subseteq B \subseteq G$ .

35. Let  $V$  and  $W$  be vector spaces of finite dimension over a field  $F$ . If  $f : V \rightarrow W$  be linear, prove that  $\dim V = \dim \text{Im } f + \dim \text{Ker } f$ .

(6 × 7 = 42 marks)

### Section D

Answer any **two** out of three questions.

Each question carries 13 marks.

36. Let  $N = a_m 10^m + a_{m-1} 10^{m-1} + \dots + a_1 10 + a_0$  be the decimal expansion of the positive integer  $N$ ;  $0 \leq a_k < 10$  and let  $S = a_0 + a_1 + \dots + a_m$ ,

Turn over

$T = a_0 - a_1 + a_2 - \dots + (-1)^m a_m$ . Prove the following :

(a)  $9 \mid N$  if and only if  $9 \mid S$ .

(b)  $11 \mid N$  if and only if  $11 \mid T$ .

(c) Use the results in (a) and (b) to show that 1571724 is divisible by both 9 and 11.

37. (a) If  $n$  is a positive integer and  $\gcd(a, n) = 1$ , prove that  $a^{\phi(n)} \equiv 1 \pmod{n}$ . Deduce that if  $p$  is a prime and  $p \nmid a$ , then  $a^{p-1} \equiv 1 \pmod{p}$ .

(b) For  $n > 1$ , prove that the sum of the positive integers less than  $n$  and relatively prime to  $n$  is  $\frac{1}{2} n \phi(n)$ .

38. Prove the following :

(a) Let  $V$  be a vector space of dimension  $n \geq 1$  over a field  $F$ . Then  $V$  is isomorphic to the vector space  $F^n$ .

(b) If  $V$  and  $W$  are vector spaces of the same dimension  $n$  over a field  $F$ , then  $V$  and  $W$  are isomorphic.

(c) A linear mapping is completely and uniquely determined by its action on a basis.

(2 × 13 = 26 marks)