D 52330 (Pages : 2) Name.....................................

Reg. No.....................................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2018

(CUCSS—PG)

Mathematics

MT 3C 14—FUNCTIONAL ANALYSIS

(2016 Syllabus Year)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *the questions.*

Each question has weightage 1.

1. Prove that the intersection of two open sets is open in a metric space.
2. State the Minkowski's inequality for measurable functions.
3. State the Riemann-Lebesgue lemma.
4. For measurable functions x and y in a set E, define $x \sim y$ if $x = y$ almost everywhere on E. Prove that $\sim$ is an equivalence relation on the set of all measurable functions on E.
5. Define quotient norm in a normed linear space.
6. In a normed linear space, prove that the sum of two open sets is open.
7. State the condition under which two norms in a linear space are equivalent.
8. State the Hahn-Banach separation theorem.
9. Give an example for a Banach space.
10. Define Schauder basis and give an example of Schauder basis for a normed space.
11. State Resonance theorem.
12. Show that a continuous map is a closed map on metric spaces.
13. State and prove the polarization identity in inner product spaces.
14. State and prove the parallelogram law in inner product spaces.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question has weightage 2.

15. Define closure of a set in a metric space. Prove that closure of a set is the smallest closed set which contains the given set.

16. For $1 \leq p < \infty$, prove that the set of all step functions on $[a, b]$ is dense in $L^p([a,b])$.

17. Let X be a normed space and Y be a closed subspace of X and $Y \neq X$. Let r be a real number such that $0 < r < 1$. Then prove that there exists some $x_r \in X$ such that $\|x_r\| = 1$ and $r \leq \text{dist}(x, Y) \leq 1$.

18. Define strictly convex space. Give an example for a strictly convex space.

19. Let X and Y be two linear spaces such that X is finite dimensional. Then prove that any linear map from X to Y is continuous.

20. Using examples, show that a linear map on a linear space may be continuous with respect to one norm and discontinuous with respect to another norm.

21. Prove that the dual X' of every normed space X is a Banach space.

22. Let X be a normed space and $P : X \longrightarrow X$ be a projection. Then prove that P is closed if and only if the subspaces R(P) and Z(P) are closed.

23. Let $\langle , \rangle$ be an inner product on a linear space X. Then prove that for all $x, y \in X, |\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$.

24. Let X be an inner product space and $\{x_1, x_2, \ldots\ldots, x_n\}$ be an orthonormal set in X. Then prove that $\|x_1 + \ldots. + x_n\|^2 = \|x_1\|^2 + \ldots\ldots + \|x_n\|^2$.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question has weightage 4.

25. If E is a measurable subset of $\mathbb{R}$, for $1 \leq p \leq \infty$, prove that the metric space $L^p(E)$ is complete.

26. State and prove the Hahn-Banach extension theorem.

27. State and prove the Uniform boundedness principle.

28. State and prove the Gram-Schmidt orthonormalization theorem in inner product space.

(2 × 4 = 8 weightage)