D 51443 (Pages : 3) Name..................................

Reg. No..............................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2018

(CUCSS—PG)

Mathematics

MT 1C 01—ALGEBRA—I

(2016 Syllabus Year)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*
Each question carries weightage 1.

1. Verify whether $\phi(x, y) = (x+1, y+2)$ is an isometry of the plane.

2. Describe a subgroup of order 4 in $\mathbb{Z}_2 \times \mathbb{Z}_6$.

3. Give two non-isomorphic groups of order 12.

4. Let G be the cyclic group $\mathbb{Z}_5$ and $X = \{1, 2, 3, 4, 5\}$ with action given by $a \cdot x = a + x \pmod 5$. Find the isotropy group G_x for $x = 1$.

5. Verify whether the series $(0) < 12\mathbb{Z} < 4\mathbb{Z} < \mathbb{Z}$ and $0 < 12\mathbb{Z} < 3\mathbb{Z} < \mathbb{Z}$ are isomorphic.

6. Find the commutator subgroup of the symmetric group S_3.

7. Verify whether $(0) < 5\mathbb{Z} < \mathbb{Z}$ is a composition series.

8. Find the number of 5 – Sylow subgroups of a group G where $|G| = 75$.

9. Let H, K be subgroups of a group G and $H \cap K = \{e\}$. Show that if $hk \in H$ for some $h \in H$ and $k \in H$ then $k = e$ where e is the identity of G.

10. Find the number of elements in the group presented as $\langle x, y : xy = y, y^2 = x \rangle$.

11. Let $\phi : \mathbb{Q}[x] \to \mathbb{R}$ be the evaluation homomorphism at $\sqrt{2}$. Find an element in Ker ϕ.

12. Verify whether $x^4 - 4$ is irreducible in $\mathbb{Q}[x]$.

Turn over

13. Find the multiplicative inverse of $(1+2i+j+k)$ in the ring of quaternions.

14. Let $\phi : \mathbb{Z}_{10} \to \mathbb{Z}_{20}$ be defined by $x \mapsto 2x$. Find $\text{Ker}\,\phi$.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question has weightage 2.

15. Verify whether $\mathbb{Z}_2 \times \mathbb{Z}_4$ is a cyclic group.

16. Describe all abelian groups of order 18.

17. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_8$ and H be the subgroup generated by (1, 1). Show that $(0,1)+H=(0,5)+H$.

18. Let G be the subgroup generated by (1, 2) in the symmetric group S_4 and X = {1, 2, 3, 4} with action given by $\sigma \cdot x = \sigma(x)$ for all $\sigma \in G$ and $x \in X$. Find the number of orbits in X.

19. Let N be a normal subgroup of a group G and H be a subgroup of G. Show that HN is a subgroup of G.

20. Describe the central series of the symmetric group S_3.

21. Show that a free group on two generators is not abelian.

22. Give a presentation of the Klein four group using two generators and verify it.

23. Verify whether $x^5 - 6x^4 + 9x + 6$ is irreducible in $\mathbb{Z}[x]$.

24. Let N be an ideal of a ring R. Show that $\phi : R \to R/N$ defined by $x \mapsto x + N$ for $x \in R$ is a homomorphism of rings with kernel N.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question has weightage 4.

25. (a) Show that if m divides the order of a finite abelian group G then G has a subgroup of order m.

(b) Let G be an abelian group of order pq where p and q are primes and $p \neq q$. Show that G is cyclic.

26. Let X be a G–set. For $g \in G$ let $\sigma_g : X \to X$ be defined by $\sigma_g(x) = g \cdot x$. Show that :

(a) σ_g is a permutation of X.

(b) $\phi : G \to S_X$ defined by $g \mapsto \sigma_g$ is a homomorphism of groups where S_X is the group of all permutations on X.

27. Let H be a subgroup of a group G and N be a normal subgroup of G. Define $\phi : HN \to H/(H \cap N)$ by $hn \mapsto h(H \cap N)$. Show that :

(a) ϕ is a well defined map.

(b) ϕ is a homomorphism onto $H/(H \cap N)$.

(c) $\text{Ker}\,\phi = N$.

28. (a) Define irreducible polynomial in $F[x]$ where F is a field.

(b) Prove that a polynomial of degree 3 in $F[x]$ is irreducible if and only if it has no zero in F.

(c) Let $f(x) \in \mathbb{Z}[x]$. Show that if $f(x)$ is irreducible in $\mathbb{Z}[x]$ then $f(x)$ considered as a polynomial in $\mathbb{Q}[x]$. is irreducible in $\mathbb{Q}[x]$.

(2 × 4 = 8 weightage)