D 31183 **(Pages : 3)** Name.......................................

Reg. No...................................

THIRD SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, NOVEMBER 2022

(CBCSS)

(November 2021 Session for SDE/Private Students)

Mathematics

MTH 3E 03—MEASURE AND INTEGRATION

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A (Short Answer Questions)

Answer **all** *questions.*
Each question has weight 1.

1. If μ is a positive measure on a σ-algebra M. Then show that $\lim \mu(A_n) = \mu(\lim A_n)$, where $\{A_n\} \in M$ with $A_1 \supset A_2 \supset A_3 .....$ and $\mu(A_1)$ is finite.
2. Show that if f is a measurable real valued function and g is continuous function, then $g \circ f$ is measurable.
3. State Riesz Representation Theorem.
4. Prove that the Lebesgue measure is translation invariant.
5. Define total variation of a complex measure μ.
6. Define absolute continuous measure, mutually perpendicular measures, and, Jordan decomposition of a measure.
7. Show that if $f \in L^1(\mu)$ and $g \in L^1(\nu)$ then $fg \in L^1(\mu \times \nu)$.
8. Show by an example that the σ-finite condition in Fubini's theorem can not be dropped to reach the conclusion of the theorem.

$(8 \times 1 = 8 \text{ weightage})$

Turn over

Part B (Short Essay)

Answer any **two** *questions from each unit.*
Each question has weightage 2.

Unit 1

9. State and prove Fatou's Lemma. Also give an example for strict inequality in this Lemma.

10. Let $f : X \to [0, \infty]$ be measurable. Then show that there exist simple functions s_n on X such that

(a) $0 \le s_1 \le s_2 \le \ldots \le f$; and (b) $s_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$.

11. Let $f : X \to [0, \infty]$ be measurable, $\varphi(E) = \int_E f d\mu \; (E \in M)$. Prove that ϕ is a measure on M and

$\int_X g d\mu = \int_X gf \, d\mu$ for every measurable real function g on X.

Unit 2

12. Prove that there exist a Lebesgue measurable set which is not a Borel set.

13. Prove that the total variation $|\mu|$ of a complex measure μ on σ-algebra M is a positive measure on M.

14. State and prove Hahn Decomposition Theorem.

Unit 3

15. Let [X, S] and $[Y, \Gamma]$ be two measurable spaces. Prove that if $E \in S \times \Gamma$, then E_x and E_y are measurable.

16. Prove that the class of elementary sets is an algebra.

17. Let m_k denote Lebesgue measure on R^k. If $k = r + s$, $r \ge 1, s \ge 1$ then m_k is the completion of the product measure $m_k \times m_s$. Prove.

(6 × 2 = 12 weightage)

Part C (Essay)

Answer any **two** *questions.*
Each question has weightage 5.

18. Suppose f and $g \in L^1(\mu)$ and α, β are complex constants. Then show that $\alpha f + \beta g \in L^1(\mu)$ and $\int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu$. Also show that $\left| \int_X f d\mu \right| \leq \int_X |f| d\mu$.

19. State and Prove Lusin's Theorem.

20. Let μ be a σ-finite positive measure on X and ϕ a bounded linear functional on $L^p(\mu), 1 \leq p < \infty$.

    Then prove that $g \in L^q(\mu)$, where q is the conjugate of p such that $\phi(f) = \int_X fg d\mu$, and $\|\phi\| = \|g\|_q$.

21. State and prove Fubini's Theorem.

(2 × 5 = 10 weightage)