C 31287 (Pages : 2) Name.................................

Reg. No.................................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2017

(CUCSS)

Mathematics

MT 3C 14—FUNCTIONAL ANALYSIS

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*

Each question carries 1 *weightage.*

1. Give an example of a metric space.
2. State the Holder's inequality for measurable functions.
3. Show that the absolute value function is a norm on the linear space of complex numbers.
4. State the Riesz lemma in normed spaces.
5. Let X and Y be normed spaces and $F : X \to Y$ be a linear map such that the range R(F) of F is finite dimensional. Then state a necessary and sufficient condition under which F is continuous.
6. Define operator norm on BL (X, Y) where X and Y are normed spaces. Show that operator norm is itself a norm.
7. Let X be a linear space over $\mathbb{C}$. Regulating X as a linear space over $\mathbb{R}$, consider a real linear functional $u : X \to \mathbb{R}$. Define $f(x) = u(x) - iu(ix), x \in X$. Prove that f is a complex-linear functional on X.
8. Let X be a normed space over K, and f be non-zero linear functional on X. If E is an open subset of X, then show that f (E) is an open subset of K.
9. Define Schauder basis for a normed space X. Write an example of Schauder basis for a normed space.
10. State the uniform boundedness principle for Banach spaces.
11. Write an example of a closed map that is not continuous.
12. State the Bessel's inequality in inner product space.
13. State the parallelogram law in an inner product space.
14. Define Hilbert space.

(14 × 1 = 14 weightage)

Turn over

Part B

Answer any **seven** *questions.*
Each question carries 2 weightage.

15. For $1 \leq p < \infty$, prove that l^p is separable.
16. Let E be a measurable subset of $\mathbb{R}$. Prove that the set of all simple measurable functions is dense in $L^\infty(E)$.
17. State and prove the Jensen's inequality in sequence spaces.
18. Let X be a normed space. If E is a convex subset of X, then prove that the closure of E is also convex.
19. Let X and Y be normed spaces with X finite dimensional. Then prove that every bijective linear map from X to Y is a homeomorphism.
20. Prove that the dual X′ of every normed space X is a Banach space.
21. Let X and Y be Banach spaces and $F \in BL(X, Y)$ be bijective. Then prove that $F^{-1} \in BL(Y, X)$.
22. Let X and Y be normed spaces and $F : X \to Y$ be linear. Then prove that F is an open map if there exists some $\gamma > 0$ such that for every $y \in Y$, there is some $x \in X$ with $F = y$ and $\|x\| \leq \gamma \|y\|$.
23. Let $\langle , \rangle$ be an inner product on a linear space X. Then prove that for all $x, y \in X, |\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle$.
24. Show that the closed graph theorem may not hold if the normed space is not a Banach space.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries 4 weightage.

25. Let X be a metric space. Prove that the intersection of a finite number of dense open subsets of X is dense in X. Further if X is complete, prove that the intersection of a countable number of dense open subsets of X is dense in X.
26. Let X be a normed space. Prove that for every subspace Y of X and every $g \in Y'$, there is a unique Hahn-Banach extension of g to X if and only if X′ is strictly convex.
27. State and prove the open mapping theorem.
28. State and prove the Gram-Schmidt orthonormalization theorem in inner product space.

(2 × 4 = 8 weightage)