

## SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2015

(CUCSS)

Mathematics

MT 2C 06—ALGEBRA—II

Time : Three Hours

Maximum : 36 Weightage

## Part A

*Short answer questions (1-14).**Answer all questions. Each question has 1 weightage.*

1. Is  $\mathbb{Q}[x]/\langle x^2 - 2 \rangle$  a field? Justify your answer.
2. Show that the polynomial  $x^2 + 1$  is irreducible in  $\mathbb{Z}_3[x]$ .
3. Is  $\mathbb{C}$  a simple extension over  $\mathbb{R}$ ? Justify your answer.
4. Find  $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]$ .
5. Let  $E$  be a finite extension of degree  $n$  over a finite field  $F$ . If  $F$  has  $q$  elements, then prove that  $E$  has  $q^n$  elements.
6. Does there exist a field of 4096 elements? Justify your answer.
7. Prove that a finite extension  $E$  of a finite field  $F$  is a simple extension of  $F$ .
8. Find all conjugates of  $3 + \sqrt{2}$  over  $\mathbb{Q}$ .
9. Find the splitting field of  $\{x^2 - 2, x^3 - 3\}$  over  $\mathbb{Q}$ .
10. Show that  $\mathbb{Q}(\sqrt[3]{2})$  has only the identity automorphism.
11. Define separable extension of a field. Give a separable extension of  $\mathbb{Q}$ .
12. Let  $p$  be a prime,  $F = \mathbb{Z}_p$  and let  $K = \text{GF}(p^{12})$ . Find  $G(K/F)$ .
13. Is regular 7-gon constructible? Justify your answer.
14. Prove that the polynomial  $x^5 - 1$  is solvable by radicals over  $\mathbb{Q}$ .

(14 × 1 = 14 weightage)

Turn over

## Part B

Answer any **seven** from the following ten questions (15-24).  
Each question has weightage 2.

15. Let  $F$  be a field. Prove that every ideal in  $F[x]$  is a principal ideal.
16. Prove that a finite extension is an algebraic extension.
17. Prove that  $\mathbb{Q}(\sqrt{3} + \sqrt{7}) = \mathbb{Q}(\sqrt{3}, \sqrt{7})$ .
18. Show that if  $E$  is a finite extension of a field  $F$  and  $[E : F]$  is a prime number, then  $E$  is a simple extension of  $F$  and  $E = F(\alpha)$  for every  $\alpha \in E$  with  $\alpha \notin F$ .
19. Let  $F$  be a field and let  $f(x)$  be irreducible in  $F[x]$ . Prove that all zeros of  $f(x)$  in  $\bar{F}$  have the same multiplicity.
20. Let  $F$  be a subfield of a field  $E$ . Prove that the set all automorphisms of  $E$  leaving  $F$  fixed forms a subgroup of the group of all automorphisms of  $E$ .
21. Show that if  $[E : F] = 2$ , then  $E$  is a splitting field over  $F$ .
22. Let  $K$  be a finite extension of  $E$  and  $E$  be a finite extension of  $F$ . Prove that  $K$  is separable over  $F$  if and only if  $K$  separable over  $E$  and  $E$  is separable over  $F$ .
23. Describe the group of polynomial  $(x^3 - 1) \in \mathbb{Q}[x]$  over  $\mathbb{Q}$ .
24. Prove that the Galois group of the  $p$ th cyclotomic extension of  $\mathbb{Q}$  for a prime  $p$  is cyclic of order  $p - 1$ .

(7 × 2 = 14 weightage)

## Part C

Answer any **two** from the following four questions (25-28).  
Each question has weightage 4.

25. (a) Prove that  $x^2 - 3$  is irreducible over  $\mathbb{Q}(\sqrt[3]{2})$ .
- (b) Let  $E$  be a finite extension field of a field  $F$  and let  $K$  be a finite extension field of  $E$ . Prove that  $K$  is a finite extension field of  $F$  and

$$[K : F] = [K : E][E : F].$$

26. Let  $F$  be a field of characteristic  $p$ . Prove that the map  $\sigma_p : F \rightarrow F$  defined by  $\sigma_p(a) = a^p$  is an automorphism. Also, prove that  $\text{Gal}(F/\mathbb{F}_p) = \mathbb{Z}_p$ .
27. Prove that every finite field is perfect.
28. Let  $F$  be a field of characteristic zero and let  $F \leq E \leq K \leq \bar{F}$ , where  $E$  is a normal extension of  $F$  and  $K$  is an extension of  $F$  by radicals. Prove that  $G(E/F)$  is a solvable group.

(2 × 4 = 8 weightage)