

C 42021

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Name.....

Reg. No.....

**FOURTH SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY)  
EXAMINATION, APRIL 2023**

(CBCSS)

Mathematics

MTH4C15—ADVANCED FUNCTIONAL ANALYSIS

(2019 Admission onwards)

Time : Three Hours

Maximum : 30 Weightage

**Part A***Answer all questions.**Each question carries a weightage 1.*

1. Define the Spectrum of a bounded operator and explain its classifications.
2. If  $X$  is infinite dimensional, show that the identity operator on  $X$  is not compact.
3. Show that the eigen vectors of a self adjoint operator corresponding to distinct eigen values are orthogonal.
4. Let  $\varphi(t) \in K[a, b]$ . Then show that there exists a sequence of polynomials  $P_n(t) \searrow \varphi(t)$  as  $n \rightarrow \infty$  for all  $t \in [a, b]$ .
5. Define closed graph operator and give an example.
6. Define Schauder basis and give an example.
7. If  $X^*$  is a separable space, then show that  $X$  is also separable.
8. Define Banach Algebra and give an example.

(8 × 1 = 8 weightage)

**Part B***Answer any two questions from each module.**Each question carries a weightage 2.*

## MODULE I

9. Let  $T$  be a compact operator on  $X$  and  $\lambda \neq 0$ . Then show that  $\Delta_\lambda$  is a closed subspace of  $X$ .

Turn over

10. Let  $H$  be a separable Hilbert space and  $T$  be a non-zero compact self-adjoint operator on  $H$ . Show that there exist an orthonormal basis consisting of eigen vectors of  $T$ .
11. Find the spectrum of the operator  $K$  on  $L^2 [0, 1]$  given by  $(K f)(t) = \int_0^1 k(t, s) f(s) ds$ , where

$$k(t, s) = \begin{cases} 1, & s \leq t \\ 0, & s > t \end{cases}$$

## MODULE II

12.  $P$  be a projection on a Hilbert space  $H$ . Show that  $P$  is self-adjoint if and only if  $\ker(P) \perp \text{Im}(P)$ .
13. Define Strong convergence of sequence of operators. Also show that strong limit of a sequence of orthoprojections is an orthoprojection.
14. Let  $P_n(t)$  and  $Q_n(t)$  be sequences of polynomials. Assume that for all  $t \in [m, M]$ ,  $Q_n(t) \searrow \psi(t) \in K$  and  $P_n(t) \searrow \phi(t) \in K$ . Let  $\psi(t) \leq \phi(t)$  for all  $t \in [m, M]$ . Then show that  $\lim_{n \rightarrow \infty} Q_n(A) =: B_1 \leq B_2 := \lim_{n \rightarrow \infty} P_n(A)$ .

## MODULE III

15. State and prove the Banach Open Mapping Theorem.
16. Define Sublinear functional and show that the *Minkowski functional* is sublinear.
17. State and prove Gelfand's theorem on maximal ideals.

(6 × 2 = 12 weightage)

## Part C

Answer any **two** questions.  
Each question carries a weightage 5.

18. (i) Let  $T$  be a compact operator on  $X$ ,  $\lambda \neq 0$ . Show that  $\Delta_\lambda = X \Rightarrow \text{Ker } T_\lambda = 0$ .
- (ii) Define *Voltera operator*  $T$  and show that  $1 \notin \sigma_p(T)$ .
19. State and prove minimax principle.

20. Show that a sequence of operators  $T_n \in L(X, Y)$  converges strongly to an operator  $T \in L(X, Y)$  if and only if:
- The sequence  $\{T_n(x)\}$  converges for any  $x$  from a dense subset of  $X$ ,
  - There exist  $C > 0$  such that  $\|T_n\| \leq C$
21. (i) Let  $K \subset X$  be a convex set. Then show that  $K$  is norm closed if and only if  $K$  is  $\omega$ -closed.
- (ii) For a real Banach space  $X$ , show that the unit ball  $\mathcal{D}(X^*) = \{f \in X^* \mid \|f\| \leq 1\}$  is a compact set in the  $\omega^*$ -topology.

(2 × 5 = 10 weightage)