

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2019

(CUCSS)

Mathematics

MT1C02—LINEAR ALGEBRA

(2016 Admissions)

Time : Three Hours

Maximum : 36 Weightage

Part A

*Answer all the questions.**Each question carries weightage of 1.*

1. Let V be a vector space over field K . Show that for any scalar $k \in K$ and $0 \in V$, $k0 = 0$.
2. Consider $V = \mathbb{R}^3$ as a vector space over \mathbb{R} . Show that W is not a subspace of V , where $W = \{(a, b, c) : a \geq 0\}$.
3. For which value of k will the vector $u = (1, -2, k)$ in \mathbb{R}^3 be a linear combination of the vectors $v = (3, 0, -2)$ and $w = (2, -1, -5)$?
4. Show that the mapping $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $F(x, y) = xy$ is not linear.
5. Let ϕ be the linear functional on \mathbb{R}^2 defined by $\phi(2, 1) = 15$ and $\phi(1, -2) = -10$. Find $\phi(x, y)$.
6. Let $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$. Find all eigenvalues and the corresponding eigenvectors of A viewed as a matrix over the real field \mathbb{R} .
7. Show that 0 is an eigenvalue of T if and only if T is singular.
8. Show that the vectors $e_1 = (1, 0, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, $e_3 = (0, 0, 1, \dots, 0)$, \dots , $e_n = (0, 0, 0, \dots, 1)$ is a basis for \mathbb{R}^n .
9. State the Cayley-Hamilton Theorem.

Turn over

10. For the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix}$, find a polynomial having the matrix A as a root.
11. Consider the three polynomials defined as follows :
 $p_1(t) = 7t^5 - 4t^2 + 3$, $p_2(t) = 2t^5 + 5t^2$, $p_3(t) = 8t^5 - 23t^2 + 6$. Is the set $\{p_1, p_2, p_3\}$ linearly independent?
12. Let \mathbb{R}^3 be equipped with the standard inner product. What is the orthogonal projection of the vector $x = (1, 2, 3)$ onto the vector $y = (3, 2, 1)$?
13. Let V be the set of polynomials with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)$. Let $f(t) = t + 2$. Find $\|f\|$.
14. If u is orthogonal to v , then show that every scalar multiple of u is also orthogonal to v .

(14 × 1 = 14 weightage)

Part B

*Answer any seven questions.
 Each question carries weightage of 2.*

15. Let V be the vector space of n -square matrices over a field \mathbb{R} . Let U and W be the subspaces of symmetric and antisymmetric matrices respectively. Show that $V = U \oplus W$.
16. Let $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the linear mapping defined by
 $T(x, y, s, t) = (x - y + s + t, x + 2s - t, x + y + 3s - 3t)$. Find a basis and the dimension of the image U of T .
17. Let V be a vector space over the field F . Prove that the intersection of any collection of subspaces of V is a subspace of V .
18. Let W be an invariant subspace for T . Show that the characteristic polynomial for the restriction operator T_W on W divides the characteristic polynomial for T .
19. State and prove the Bessel's inequality.

20. Find a unit vector orthogonal to $v_1 = (1, 1, 2)$ and $v_2 = (0, 1, 3)$ in \mathbb{R}^3 .
21. For what value of k , $\langle u, v \rangle = x_1y_1 - 3x_1y_2 - 3x_2y_1 + kx_2y_2$, where $u = (x_1, x_2)$, $v = (y_1, y_2)$ is an inner product in \mathbb{R}^2 ?
22. State and Prove Cauchy-Schwarz inequality.
23. Verify that the following is an inner product in \mathbb{R}^2 : $\langle u, v \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$, where $u = (x_1, x_2)$, $v = (y_1, y_2)$.
24. If $\lambda \in \mathbf{F}$ is a characteristic value of a linear operator T on a vector space V , then show that for any polynomial $f(x)$ over \mathbf{F} , $f(\lambda)$ is a characteristic value of $f(T)$.

(7 × 2 = 14 weightage)

Part C*Answer any two questions.**Each question carries weightage of 4.*

25. If W_1 and W_2 are finite dimensional subspaces of a vector space V , then prove that $W_1 + W_2$ is finite dimensional and $\dim W_1 + \dim W_2 = \dim (W_1 \cap W_2) + \dim (W_1 + W_2)$.
26. Prove that a linear operator $T: V \rightarrow V$ has a diagonal matrix representation if and only if its minimal polynomial $m(t)$ is a product of distinct linear polynomials.
27. Let V and W be vector space over the field \mathbf{F} , and let T be a linear transformation from V into W . The null space of T^t (transpose of T) is the annihilator of the range of T . If V and W are finite dimensional, then prove that (i) $\text{rank}(T^t) = \text{rank}(T)$, (ii) the range of (T^t) is the annihilator of the null space of T .
28. Apply Gram-Schmidt process to the vectors $w_1 = (1, 0, 1, 0)$, $w_2 = (1, 1, 1, 1)$ and $w_3 = (0, 1, 2, 1)$ to compute an orthonormal set in \mathbb{R}^4 .

(2 × 4 = 8 weightage)