

D 140688**(Pages : 3)****Name.....****Reg. No.....**

**SECOND SEMESTER M.Sc. (CBCSS) REGULAR/SUPPLEMENTARY DEGREE
EXAMINATION, APRIL 2026**

Mathematics

MTH 2C 07—REAL ANALYSIS—II

(2019 Admission onwards)

Time : Three Hours

Maximum : 30 Weightage

Part A

Answer all questions.

Each question carries a weightage of 1.

1. Verify whether the set $\mathbb{N}$ of natural numbers is a Borel set.
2. Find $m(\mathbb{Q})$ where m is the Lebesgue measure and $\mathbb{Q}$ is the set of all rationals.
3. Let $f(x)$ be defined on the reals by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is an integer} \\ 0 & \text{otherwise.} \end{cases}$$

Verify whether f is Lebesgue measurable.

4. Let

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{if } 1 \leq x \leq 2. \end{cases}$$

Find $\int_{[0,2]} f$.

5. Let $f(x)$ in $[0, 1]$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{otherwise.} \end{cases}$$

Find $\int_E f$ where E is the set of all rationals in $[0, 1]$.

6. Let $F = \{f_1, f_2, \dots, f_n\}$ be a family of integrable functions on $[0, 1]$. Verify whether F is uniformly integrable in $[0, 1]$.
7. Show that every increasing function on $[0, 1]$ is of bounded variation.
8. Let $f(x) = x^2$ in $[0, 1]$. Show that $f(x)$ is absolutely continuous.

(8 × 1 = 8 weightage)

Turn Over

Part B

*Answer any two questions from each module.
Each question carries a weightage of 2.*

Module I

9. Let $\mathbb{N}$ be the set of all natural numbers. Show that $m^*(\mathbb{N}) = 0$ where m^* is the Lebesgue outer measure.
10. Show that if A and B are measurable sets then $A \cup B$ is measurable.
11. Show that if (B_n) is a descending chain of Lebesgue measurable sets of finite measure then $m(\cap_n B_n) = \lim_{n \rightarrow \infty} m(B_n)$.

Module II

12. Let E be a measurable set of finite measure and ϕ, ψ be simple functions on E . Show that $\int_E (\phi - \psi) = \int_E \phi - \int_E \psi$.
13. Let f be a bounded measurable function on a set E of finite measure. Show that $|\int_E f| \leq \int_E |f|$.
14. Let E be a measurable set. Let (f_n) be a sequence of non-negative integrable functions on E such that $\lim_{n \rightarrow \infty} \int_E f_n = 0$. Show that (f_n) converges to 0 in measure.

Module III

15. Let f be a function of bounded variation on $[0, 1]$. Show that f is differentiable a.e. on $(0, 1)$.
16. Let f be absolutely continuous on $[0, 1]$. Show that f is of bounded variation on $[0, 1]$.
17. Show that every rapidly Cauchy sequence in a normed linear space X is a Cauchy sequence in X .

(6 × 2 = 12 weightage)

Part C

*Answer any two questions.
Each question carries a weightage of 5.*

18. (a) Define Lebesgue measurable set.
(b) Show that a subset E of $\mathbb{R}$ is measurable if and only if for each $\varepsilon > 0$ there exists an open set O containing E such that $m^*(O \setminus E) < \varepsilon$ where m^* is the Lebesgue outer measure on $\mathbb{R}$.
19. (a) Let (f_n) be a sequence of measurable functions on a measurable set E such that $f_n \rightarrow f$ pointwise on E . Show that f is measurable.
(b) Show that the characteristic function χ_A is measurable if and only if A is measurable.

20. (a) Define uniformly integrable family of functions.
- (b) Let (f_n) be a sequence of functions which is uniformly integrable over a set E of finite measure. Show that if $f_n \rightarrow f$ pointwise a.e. on E then f is integrable on E .
21. (a) Let f be a monotone function on an open interval (a, b) . Show that f is continuous except possibly at a countable number of points in (a, b) .
- (b) Let f be an increasing function on $[a, b]$. Show that f' is integrable over $[a, b]$ and
- $$\int_a^b f' \leq f(b) - f(a).$$

(2 × 5 = 10 weightage)