

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2018

(CUCSS—PG)

Mathematics

MT 3C 13—COMPLEX ANALYSIS

(2016 Syllabus Year)

Time : Three Hours

Maximum : 36 Weightage

Part A

*Answer all questions.**Each question carries a weightage of 1.*

1. If $f : \Omega \rightarrow \mathbb{C}$ is analytic (where Ω is a region) and if $f'(z) \neq 0$ for any z , then prove that f is conformal.
2. Prove that the reflection $z \rightarrow \bar{z}$ is not a linear transformation.
3. Prove that a general linear transformation is the composition of translations, an inversion, a rotation and a homothetic transformation.
4. Compute $\int_{\gamma} x dz$ where γ is the directed line segment from 0 to $1 + i$.
5. State the Maximum principle.
6. Prove that a function which is analytic and bounded in the whole plane is a constant.
7. Show that the function $\sin z$ has an essential singularity at ∞ .
8. Find the poles and residues of the function $\frac{1}{(z^2 - 1)^2}$.
9. Prove that $u = y^3 - 3x^2y$ is a harmonic function.
10. Prove the Mean-Value property for harmonic functions.

Turn over

11. Develop $\log \left(\frac{\sin z}{z} \right)$ in powers of z upto the term z^6 .
12. Prove that the sum of residues of an elliptic function is zero.
13. Find the Laurent series for $\frac{1}{(z-1)(z-2)}$ valid in the annulus $1 < |z| < 2$.
14. Briefly describe unimodular transformation.

(14 × 1 = 14 weightage)

Part B

*Answer any seven questions.
Each question carries a weightage of 2.*

15. Prove that the principal branch of the logarithm function is analytic.
16. Prove that the cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a circle or on a straight line.
17. State and prove Morera's theorem.
18. Show that a function which is analytic in the whole plane and has a non-essential singularity at ∞ reduces to a polynomial.
19. If $f(z)$ is meromorphic in Ω with the zeros a_j and poles b_k then prove that :

$$\frac{1}{2\pi i} \int_r \frac{f'(z)}{f(z)} = \sum_j n(r, a_j) - \sum_k n(r, b_k)$$

for every cycle r which is homologous to zero in Ω and does not pass through any of the zeros or poles.

20. Evaluate $\int_0^{\infty} \frac{x^2}{x^4 + 5x^2 + 6} dx$

by the method of residues.

21. How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the disk $|z| < 1$.

22. If the functions $f_n(z)$ are analytic and $\neq 0$ in a region Ω , and if $f_n(z)$ converges to $f(z)$, uniformly on every compact subset of Ω , then prove that $f(z)$ is either identically zero or never equal to zero in Ω .

23. Prove that the zeros a_1, \dots, a_n and poles b_1, b_2, \dots, b_n of an elliptic function satisfy :

$$a_1 + \dots + a_n \equiv b_1 + \dots + b_n \pmod{M}.$$

24. Show that any elliptic function with periods w_1, w_2 can be written as :

$$C \prod_{k=1}^n \frac{\sigma(z - a_k)}{\sigma(z - b_k)} \quad (C = \text{constant}).$$

(7 × 2 = 14 weightage)

Part C

*Answer any two questions.
Each question carries a weightage of 4.*

25. Let $f(z)$ be analytic on the set R^1 obtained from a rectangle R by omitting a finite number of interior points G_j . If $\lim_{z \rightarrow G_j} (z - G_j) f(z) = 0$ for all j , then prove that :

$$\int_{\partial R} f(z) dz = 0.$$

26. State and prove the lemma of Schwarz.

Turn over

27. Describe the Laurent series development.
28. For the Weierstrass P-function prove that :

$$(a) \quad P(2z) = \frac{1}{4} \left(\frac{P''(z)}{P'(z)} \right)^2 - 2P(z)$$

$$(b) \quad P'(z) = -\sigma(2z) / (\sigma(z))^4.$$

(2 × 4 = 8 weightage)