C 31285 (Pages : 3) Name..............................

Reg. No..............................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2017

(CUCSS)

Mathematics

MT 3C 12—MULTIVARIATE CALCULUS AND GEOMETRY

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A (Short Answer Questions)

Answer **all** *questions.*

Each question carries 1 *weightage.*

1. If A is a linear transformation from the vector space $\mathbb{R}^n$ to the vector space $\mathbb{R}^m$, then prove that A is uniformly continuous.

2. If A is a linear transformation from the vector space $\mathbb{R}^n$ to the vector space $\mathbb{R}^m$ and B is a linear transformation from the vector space $\mathbb{R}^m$ to the vector space $\mathbb{R}^k$, then prove that :

$$\|BA\| \leq \|B\| \, \|A\|.$$

3. If A is a linear transformation from the vector space $\mathbb{R}^n$ to the vector space $\mathbb{R}^m$ and if $x \in \mathbb{R}^n$, then prove that $A'(x) = A$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = 2x^3 - 3x^2 + 2y^3 + 3y^2$. Find the gradient of f at (1, – 1).

5. Is $\gamma(t) = \left(t^2, t^4\right)$ a parametrization of parabola ? Justify your answer.

6. Find the tangent vector of the parametrised curve $\gamma(t) = \left(e^t, t^2\right)$.

7. Calculate the arc-length of the logarithmic spiral $\gamma(t) = \left(e^t \cos t, e^t \sin t\right)$ starting at the point (1, 0).

8. Compute the curvature of the curve $\gamma(t) = \left(\cos^3 t, \sin^3 t\right)$.

9. Prove that curvature of a straight line is zero.

10. Prove that an open disc in the *xy*-plane is a surface.

11. Define closed curve and give an example of it.

12. Find the first fundamental form of the surface $\sigma(u, v) = \left(u - v, u + v, u^2 + v^2\right)$.

Turn over

13. Find the second fundamental form of the elliptic paraboloid $\sigma(u, v) = \left(u - v, u + v, u^2 + v^2\right)$.

14. Define Weingarten map.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question carries 2 weightage.

15. Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then prove that dim $X \le r$.

16. Let A be a linear operator on a finite dimensional vector space X. Prove that range of A is X if and only if $A(x) = 0$ implies $x = 0$ for all $x \in X$.

17. Prove that $d(A, B) = \|A - B\|$ is a metric on the set of all linear transformations from $\mathbb{R}^n$ into $\mathbb{R}^m$.

18. Prove that any parametrization of a regular curve is regular.

19. Find the torsion of a circular helix $\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta)$.

20. Let $\sigma : U \to \mathbb{R}^3$ be a patch of a surface $\mathcal{S}$ containing a point $p \in \mathcal{S}$ and let (u, v) be co-ordinates in U. Prove that the tangent space of $\mathcal{S}$ at p is the vector subspace of $\mathbb{R}^3$ spanned by the vectors σ_u and σ_v.

21. Let $f : \mathcal{S} \to \mathcal{S}'$ be a smooth map between surfaces and $p \in \mathcal{S}$. Prove that the derivative $D_p f : T_p\mathcal{S} \to T_{f(p)}\mathcal{S}'$ is a linear map.

22. With the usual notations prove that the mean curvature is

$$H = \frac{LG - 2MF + NE}{2\left(EG - F^2\right)}.$$

23. If γ is a unit speed curve on an oriented surface $\mathcal{S}$, then prove that its normal curvature is $K_n = \langle \dot{r}, \dot{r} \rangle$.

24. Prove that Gaussian curvature of a ruled surface is negative or zero.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries 4 weightage.

25. Let $E \subset \mathbb{R}^n$ be an open set and the map $f : E \to \mathbb{R}^k$ be differentiable at $x_0 \in E$. If g maps an open set containing $f(E)$ into $\mathbb{R}^m$ and g is differentiable at $f(x_0)$, then prove that the map $F : E \to \mathbb{R}^m$ defined by $F(x) = g(f(x))$ is differentiable at x_0 and $F'(x_0) = g'(f(x_0))f'(x_0)$.

26. Let X be a complete metric space and φ be a contraction mapping of X into X. Prove that there exists one and only one $x \in X$ with $\varphi(x) = x$.

27. Let γ be a unit speed curve in $\mathbb{R}^3$ with constant curvature and zero torsion. Prove that γ is part of a circle.

28. Let $\sigma : U \to \mathbb{R}^3$ be a surface patch, let $(u_0, v_0) \in U$, and let $\delta > 0$ be such that the closed disc

$$R_\delta = \left\{(u, v) \in \mathbb{R}^2 : (u - u_0)^2 + (v - v_0)^2 \leq \delta^2\right\}$$

with centre (u_0, v_0) and radius δ is contained in U. Prove that

$$\lim_{\delta \to 0} \frac{A_N(R_\delta)}{A_N(R_\delta)} = |K|,$$

where K is the Gaussian curvature of σ at $\sigma(u_0, v_0)$.

(2 × 4 = 8 weightage)