D 72970 (Pages : 3) Name..................................

Reg. No..................................

FIRST SEMESTER M.A./M.Sc./M.Com. DEGREE EXAMINATION DECEMBER 2019

(CBCSS)

Mathematics

MTH 1C 02—LINEAR ALGEBRA

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *the questions.*

Each question carries 1 *weightage.*

1. Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector α in V there are unique vectors α_1 in W_1 and α_2 in W_2 such that $\alpha = \alpha_1 + \alpha_2$.

2. Is there a linear transformation T from R^3 into R^2 such that $T(1,-1,1) = (1,0)$ and $T(1,1,1) = (0,1)$? Justify your answer.

3. If W_1 and W_2 are subspaces of a finite-dimensional vector space, then prove that $W_1 = W_2$ iff $W_1^{\circ} = W_2^{\circ}$.

4. Define minimal polynomial for a $n \times n$ matrix. Find a 3×3 matrix for which the minimal polynomial is x^2.

5. Let T be the linear operator on R^2 which is represented in the standard ordered basis by the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Show that the only subspaces of R^2 which are invariant under T are R^2 and the zero space.

6. Find a projection E which projects R^2 onto the subspace spanned by $(1,-1)$ along the subspace spanned by $(1,2)$.

Turn over

7. Let V be an inner product space and let $\alpha, \beta \in V$. Show that $|(\alpha|\beta)| \leq \|\alpha\| \cdot \|\beta\|$.

8. Let V be an inner product space, W a finite-dimensional subspace, and E the orthogonal projection of V on W. Show that the mapping $\beta \mapsto \beta - E\beta$ is the orthogonal projection of V onto $W^{\perp}$.

(8 × 1 = 8 weightage)

Part B

Answer any **two** *questions from each of the following units.*
Each question carries 2 weightage.

Unit I

9. Show that the subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of vectors in S.

10. Let V be the vector space of all 2 × 2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V.

11. Let V and W be vector spaces over the field F and let u be an isomorphism of V onto W. Prove that the mapping $T \mapsto uTu^{-1}$ is an isomorphism of L (V, V) onto L (W, W).

Unit II

12. Let T be the linear operator on R^3 defined by $T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$. What is the matrix of T in the standard ordered basis for R^3 ?

13. Let T be a linear operator on an n-dimensional vector space V. Show that the characteristic and minimal polynomials for T have the same roots, except for multiplicities.

14. Let A be any $m \times n$ matrix over the field F. Show that the row rank of A is equal to the column rank of A.

Unit III

15. Let V be a real vector space and E an idempotent linear operator on V. Prove that $I + E$ is invertible. Find $(I + E)^{-1}$.

16. Apply the Gram-Schmidt process to the vectors $\beta_1 = (1, 0, 1)$, $\beta_2 = (1, 0, -1)$, $\beta_3 = (0, 3, 4)$, to obtain an orthonormal basis for R^3 with the standard inner product.

17. State and prove Bessel's inequality.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries 5 weightage.

18. Let V be an n-dimensional vector space over the field F, and let β and β' be two ordered basis of V. Show that there is a unique invertible $n \times n$ matrix P with entries in F such that :

    (i) $[\alpha]_\beta = P[\alpha]_{\beta'}$; (ii) $[\alpha]_{\beta'} = P^{-1}[\alpha]_\beta$ for every vector α in V.

19. Let V and W be vector spaces over the filed F and let T be a linear transformation from V into W. If V is finite-dimensional, then show that

    rank (T) + nullity (T) = dim V.

20. (a) Let $g, f_1, \ldots\ldots, f_r$ be linear funtionals on a vector space V with respective null spaces $N, N_1, \ldots\ldots, N_r$. Show that g is a linear combination of $f_1, \ldots\ldots, f_r$ iff N contains the intersection $N_1 \cap \ldots\ldots \cap N_r$.

    (b) If W is a subspace of a finite-dimensional vector space V and if $\{g_1, \ldots\ldots, g_r\}$ is any basis for W°, prove that $W = \bigcap_{i=1}^{r} N_{g_i}$.

21. Let T be a linear operator on a finite-dimensional space V. Show that if T is diagonalizable and if $c_1, \ldots\ldots, c_k$ are the distinct characteristic values of T, then there exist linear operators $E_1, E_2, \ldots, E_k$ on V such that :

    (i) $T = c_1E_1 + \ldots. + c_kE_k$.

    (ii) $I = E_1 + E_2 + \ldots.. + E_k$

    (iii) $E_i\, E_j = 0, \quad i \neq j$

    (iv) $E_i^2 = E_i$.

    (v) The range of E_i is the characteristic space for T associated with c_i.

(2 × 5 = 10 weightage)