

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2017

(CUCSS)

Mathematics

MT 4E 02—ALGEBRAIC NUMBER THEORY

Time : Three Hours

Maximum : 36 Weightage

*Standard notation as in the prescribed text is followed.***Part A***Answer all questions.**Each question carries 1 weightage.*

1. Show that \mathbb{Q} is not a cyclic group.
2. Find the order of the group G/H where G is free abelian with \mathbb{Z} -basis, x, y, z and H is generated by $2x, 3y, 7z$.
3. Find all monomorphisms $\mathbb{Q}(\sqrt[3]{7}) \rightarrow \mathbb{C}$.
4. Compute integral basis and discriminant for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.
5. Let $K = \mathbb{Q}(\xi)$ where $\xi = e^{\frac{2\pi i}{p}}$ for a rational prime p . In the ring of integers $\mathbb{Z}[\xi]$, show that $\alpha \in \mathbb{Z}[\xi]$ is a unit iff $N_k(\alpha) = \pm 1$.
6. Which of the following elements of $\mathbb{Z}[i]$ are irreducible? $1 + i, 5, 12i$.
7. Find a ring which is not noetherian.
8. Show that a prime in a domain D is always irreducible.
9. Let R be a ring and \mathfrak{a} a maximal ideal of R . Show that R/\mathfrak{a} is a field.
10. Find all fractional ideals of $\mathbb{Z}[i]$.
11. Sketch the lattice in \mathbb{R}^2 generated by $(1, 1)$ and $(2, 3)$ and find a fundamental domain for the lattice.

Turn over

12. State Minkowski's theorem.
13. Let L be an n -dimensional lattice in \mathbb{R}^n with basis $\{e_1, e_2, \dots, e_n\}$. Suppose $e = (a_1, \dots, a_n)$. Show that the volume of the fundamental domain T of L defined by this basis is $v(T) = |\det a_{ij}|$.
14. Define regular prime and show that 5 is a regular prime.

(14 × 1 = 14 weightage)

Part B*Answer any seven questions.**Each question carries 2 weightage.*

15. Let R be a ring. Show that every symmetric polynomial in $\mathbb{R}[t_1, t_2, \dots, t_n]$ is expressible as a polynomial with coefficients in \mathbb{R} in the elementary symmetric polynomials s_1, s_2, \dots, s_n .
16. Show that if K is a number field then $K = \mathbb{Q}(\theta)$ for some algebraic number θ .
17. Show that every number field K possesses an integral basis, and the additive group of the ring of integers is a free abelian group of rank n , where n is the degree of K .
18. Show that the discriminant of $\mathbb{Q}(\xi)$, where $\xi = e^{\frac{2\pi i}{p}}$ and p is an odd prime, is $(-1)^{\frac{p-1}{2}} \cdot p^{p-2}$.
19. Show that an integral domain D is noetherian iff D satisfies the ascending chain condition.
20. Show that every principal ideal domain is a unique factorization domain.
21. Show that if a, b are non-zero ideals of the ring of integers D of a number field K , then there exists $\alpha \in a$ such that $\alpha a^{-1} + b = D$.
22. Let $K = \mathbb{Q}(\theta)$ be a number field of degree n , where θ is an algebraic integer. Show that if $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis for K over \mathbb{Q} then $\sigma(\alpha_1), \sigma(\alpha_2), \dots, \sigma(\alpha_n)$ are linearly independent over \mathbb{R} , where σ is the usual σ -map.
23. Determine the class number of $\mathbb{Q}(\sqrt{-3})$.
24. Let $K = \mathbb{Q}(\xi)$, where $\xi = e^{\frac{2\pi i}{p}}$ for an odd prime p . Show that the only roots of unity in K are $\pm \xi^s$ for integers s .

(7 × 2 = 14 weightage)

Part C

Answer any two questions.

Each question carries 4 weightage.

25. Show that every subgroup H of a free abelian group G of rank n is free of rank $s \leq n$.
26. Show that in a domain in which factorization into irreducibles is possible, factorization is unique iff every irreducible is prime.
27. Factorize the ideal $\langle 18 \rangle$ in $\mathbb{Z}[\sqrt{-17}]$.
28. Show that an additive subgroup of \mathbb{R}^n is a lattice iff it is discrete.

(2 × 4 = 8 weightage)