D 32716 (Pages : 2) Name..............................

Reg. No..............................

FIRST SEMESTER M.Sc. (CBCSS) REGULAR/SUPPLEMENTARY DEGREE EXAMINATION, NOVEMBER 2022

Mathematics

MTH1C01—ALGEBRA—I

(2019 Admission onwards)

Time : Three Hours Maximum Weightage : 30

Part A

Answer **all** *questions.*
Each question carries a weightage 1.

1. Do the rotatioins, together with the identity map, form a subgroup of the group of plane isometries ? Why or why not ?
2. Find the order of (3, 6, 12, 16) in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{20} \times \mathbb{Z}_{24}$.
3. Find the order of the factor group $(\mathbb{Z}_{12} \times \mathbb{Z}_{18})/\langle(4,3)\rangle$.
4. In the group $\mathbb{Z}_{36}$ with $H = \langle 6 \rangle$ and $N = \langle 9 \rangle$. List the elements in HN. List the cosets in HN/N, showing the elements in each coset.
5. Show that no group of order 36 is simple.
6. How many different homomorphisms are there of a free group of rank 2 onto $\mathbb{Z}_4$?
7. Give a presentation of $\mathbb{Z}_4$ involving one generator, involving two generators; involving three generators.
8. The polynomial $x^4 + 4$ can be factored into linear factors $\mathbb{Z}_5[x]$. Find this factorization.

(8 × 1 = 8 weightage)

Part B

Answer any **six** *questions choosing* **two** *from each unit.*
Each question carries a weightage 2.

Unit 1

9. If m divides the order of a finite abelian group G, then show that G has a subgroup of order m.
10. Let H be a normal subgroup of G. Show that the cosets of H form a group G/H under the binary operation $(a\text{H})(b\text{H}) = (ab)\text{H}$.

Turn over

11. Let X be a G-set and let $x \in X$. Then $|Gx| = (G : G_x)$. If $|G|$ is finite, then show that $|Gx|$ is a divisor of $|G|$.

Unit 2

12. Let H be a subgroup of G and let N be a normal subgroup of G. Prove that $(HN)/N \simeq H/(H \cap N)$.

13. Let P_1 and P_2 be Sylow p-subgroups of a finite group G. Prove that P_1 and P_2 are conjugate subgroups of G.

14. For a prime number p, prove that every group G of order p^2 is abelian.

Unit 3

15. Compute the evaluation homormorphism $\phi_5\left[(x^3+2)(4x^2+3)(x^7+3x^2+1)\right]$, $F = E = \mathbb{Z}_7$.

16. Let $f(x) \in F[x]$, and let $f(x)$ be of degree 2 or 3. Prove that $f(x)$ is reducible over F if and only if it has a zero in F.

17. Let $G = \{e, a, b\}$ be a cyclic group of order 3 with identity element e. Write the element $(3e+3a+3b)^4$ in the group algebra $\mathbb{Z}_5G$ in the form $re + sa + tb$ for $r, s, t \in \mathbb{Z}_5$.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage 5.

18. (a) Prove that a factor group of a cyclic group is cyclic.

    (b) Let G be a group. The set of all commutators $aba^{-1}b^{-1}$ for $a, b \in G$ generates a subgroup C (the commutator subgroup) of G. Show that the subgroup C is a normal subgroup of G, if N is a normal subgroup of G, then show that G/N is abelian if and only if $C \leq N$.

19. (a) Prove that M is a maximal normal subgroup of G if and only if G/M is simple.

    (b) Let G be the additive group of real numbers. Let the action of $\theta \in G$ on the real plane $\mathbb{R}^2$ be given by rotating the plane counterclockwise about the origin through θ radians. Let P be a point other than the origin in the plane. Show $\mathbb{R}^2$ is a G-set. Describe geometrically the orbit containing P. Find the group G_P.

20. (a) State and prove First Sylow Theorem.

    (b) Prove that the center of a finite nontrivial p-group G is nontrivial.

21. (a) State and prove Division Algorithm for $F[x]$.

    (b) Demonstrate that $x^3 + 3x^2 - 8$ is irreducible over $\mathbb{Q}$.

(2 × 5 = 10 weightage)