D 42539 (Pages : 2) Name..............................

Reg. No..............................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2018

(CUCSS)

Mathematics

Group I : MT 4E 02—ALGEBRAIC NUMBER THEORY

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*
Each question carries weightage 1.

1. Express the polynomial $t_1^2 + t_2^2 + t_3^2$ $(n = 3)$ in terms of elementary symmetric polynomials.
2. Express $Q(\sqrt{3}, \sqrt[3]{5})$ in the form $Q(\theta)$.
3. If $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis of a number field K consisting of integers, then show that the discriminant $\Delta[\alpha_1, \ldots, \alpha_n]$ is a rational integer, not equal to zero.
4. Find integral basis and discriminant for $Q(\sqrt{6})$.
5. Let $K = Q(\xi)$ where $\xi = e^{2\pi i/5}$. Calculate $N_K(\alpha)$ and $T_K(\alpha)$ for $\alpha = \xi^2$.
6. Show that the ring of integers in a number field K is noetherian.
7. Show that a prime in a domain D is always irreducible.
8. Determine the group of units of the ring of integers of $Q(\sqrt{-1})$.
9. Let R be a ring and a an ideal of R. Show that a is maximal iff $\frac{R}{a}$ is a field.
10. If $p = \langle 3, 1 - \sqrt{-5} \rangle$ is the ideal in $Z[-5]$, then determine pp^{-1}.
11. Sketch the lattice in R^2 generated by (–2, –7) and (4, –3).
12. State Minkowski's theorem.
13. Let L be an n-dimensional lattice in R^n with basis $\{e_1, e_2, \ldots, e_n\}$. Suppose $e_i = (a_{1i}, a_{2i}, \ldots, a_{ni})$. Show that the volume of the fundamental T of L defined by this basis is $v(T) = |\det a_{ij}|$.
14. Factorize the ideal $\langle 3 \rangle$ in the ring of integers of $Q(\sqrt{5})$.

(14 × 1 = 14 weightage)

Turn over

Part B

Answer any **seven** *questions.*
Each question carries weightage 2.

15. Show that every finitely generated abelian group with n generators is the direct product of a finite abelian group and a free group on k generators where $k \le n$.

16. Show that the discriminant of any basis for $k = Q(\theta)$ is rational and non-zero.

17. Let θ be a complex number satisfying a monic polynomial equation whose coefficients are algebraic integers. Show that θ is an algebraic integer.

18. Let $k = Q(\xi)$ where $\xi = e^{2\pi i/p}$ for a rational prime p. In the ring of integers $Z[\xi]$, show that $\alpha \varepsilon Z[\xi]$ is a unit if and only if $N_k(\alpha) = \pm 1$.

19. Show that factorization into irreducibles is not unique in the ring of $Q\left(\sqrt{15}\right)$.

20. Show that every principal ideal domain is a unique factorization domain.

21. Show that if a, b are non-zero ideals of the ring of integers D of a number field K of degree n, then there exists $\alpha \varepsilon a$ such that $\alpha a^{-1} + b = D$.

22. With usual notations, show that if $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a basis for K over Q then $\sigma(\alpha_1), \ldots, \sigma(\alpha_n)$ are linearly independent over R.

23. Show that the class-group of a number field is a finite abelian group.

24. Show that the equation $x^4 + y^4 = z^2$ has no integer solutions with $x, y, z \ne 0$.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries weightage 4.

25. Show that if K is a number field then $k = Q(\theta)$ for some algebraic number θ.

26. Show that the ring D of integers of $Q[\xi]$ is $Z[\xi]$.

27. Show that the ring of integers D of $Q(\sqrt{-7})$ is an Euclidean domain.

28. Show that an additive subgroup of R^n is a lattice if and only if it is discrete.

(2 × 4 = 8 weightage)