C 83063 (Pages : 3) Name..................................

Reg. No..................................

SECOND SEMESTER M.A./M.Sc./M.Com. DEGREE EXAMINATION, JUNE 2020

(CBCSS)

Mathematics

MT 2C 07—REAL ANALYSIS II

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*

***Each question carries a weightage of* 1.**

1. Prove that a countable set has outer measure zero.
2. Show that the translate of a measurable set is measurable.
3. Define a measurable function. Give an example of a measurable function.
4. Let f be a bounded measurable function on a set of finite measure E. If A and B are disjoint measurable subsets of E, then prove that

   $$\int_A f + \int_B f = \int_{A \cup B} f$$

5. For a number α, define $f(x) = x^{\alpha}$ for $0 < x \leq 1$ and $f(0) = 0$. Compute $\int_0^1 f$.
6. Let f be integrable over E. Prove that for each $\epsilon > 0$, there is a Set of finite measure E_0 for which $\int_{E-E_0} |f| < \epsilon$.
7. Find the upper and lower derivatives of the function f defined on $\mathbb{R}$ by

   $$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

8. For $1 < p < \infty, q$ the conjugate of p, and any two positive numbers a and b, prove that $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

(8 × 1 = 8 weightage)

Turn over

Part B

Answer any **six** *questions by choosing two questions from each unit.*
Each question carries a weightage of 2.

Unit I

9. Show that the union of a finite collection of measurable sets is measurable.

10. If E_1 and E_2 are measurable, then prove that

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

11. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to the function *f*. Prove that *f* is measurable.

Unit II

12. Let *f* and *g* be bounded measurable functions on a set of finite measure E. Prove that

$$\int_E (f+g) = \int_E f + \int_E g.$$

13. Let *f* be integrable over E and $\{E_n\}_{n=1}^{\infty}$ a disjoint countable collection of measurable functions of E whose union is E. Prove that $\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f.$

14. Show that linear combinations of sequences that converge in measure on a set of finite measure also converge in measure.

Unit III

15. Prove that a function *f* on a closed bounded interval $[a,b]$ is absolutely continuous on $[a,b]$ if and only if it is an indefinite integral over $[a,b]$.

16. If ϕ is differentiable on (a, b) and its derivative ϕ' is increasing, then prove that ϕ is convex.

17. State and prove Minkowski's Inequality.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. (a) Prove that any set of outer measure zero is measurable.

(b) Prove that the outer measure of an interval is its length.

19. (a) State and prove Fatou's Lemma.

(b) Let E be of finite measure. Suppose the sequence of functions $\{f_n\}$ is uniformly integrable over E. If $\{f_n\} \to f$ pointwise a.e. on E. then prove that f is integrable over E and $\lim_{n\to\infty} \int_E |f_n| = \int_E f$.

20. Let f be a bounded function on the closed, bounded interval $[a,b]$. Prove that f is Riemann integrable over $[a,b]$ if and only if the set of points in $[a,b]$ at which f fails to be continuous has measure zero.

21. (a) Let the function f be absolutely continuous on the closed bounded interval $[a,b]$. Prove that f is the difference of increasing absolutely continuous functions and, in particular is of bounded variation.

(b) Let E be a measurable set and $1 \le p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to the function f which belongs to $L^p(E)$. Prove that $\{f_n\} \to f$ in $L^p(E)$ if and only if $\lim_{n\to\infty} \int_E |f_n|^p = \int_E |f|^p$.

(2 × 5 = 10 weightage)