C 31286 (Pages : 3) Name.....

Reg. No.....

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2017

(CUCSS)

Mathematics

MT 3C 13—COMPLEX ANALYSIS

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*
Each question carries 1 *weightage.*

1. Prove that an analytic function defined in a region Ω whose modulus is constant is a constant function.
2. Prove that a general linear transformation is the composition of translations, an inversion, a rotation and a homothetic transformation.
3. Find the linear transformation which carries $o, i, -i$ into $1, -1, 0$.
4. If the piecewise differentiable closed curve r does not pass through the point a, prove that the value of the integral $\int_r \frac{dz}{z-a}$ is a multiple of $2\pi i$.
5. If $P(z)$ is a non-constant polynomial in $\mathbb{C}$, then prove that $P(z)$ has a root in $\mathbb{C}$.
6. Show that the function e^z has an essential singularity at ∞.
7. State Rouche's theorem.
8. Evaluate the residue of $\frac{z^2}{(z-1)(z-2)(z-3)}$ at $z = 2$.
9. If u is harmonic, prove that $\frac{\partial u}{\partial x}$ is also harmonic.
10. Prove that the arithmetic mean of a harmonic function over concentric circles $|z| = r$ is a linear function of $\log r$.
11. Develop $\tan z$ in powers of z upto the term containing z^5.
12. Find the Laurent expansion of $f(z) = \frac{1}{z(z-1)(z-2)}$ in the annulus $1 < |z| < 2$.

Turn over

13. Prove that an elliptic function without poles is a constant.

14. Prove that a non-constant elliptic function has equally many poles as it has zeros.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question carries 2 *weightage.*

15. Prove that the cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a circle or on a straight line.

16. State and prove symmetry principle.

17. Let r be a piecewise differentiable closed curve and suppose that r does not pass through the point a. Prove that, as a function of a, the index $n(r, a)$ is a constant in each of the regions determined by r.

18. Define pole and essential singularity, giving one example of each.

19. Prove that an analytic function comes arbitrarily close to any complex value in every neighbourhood of an essential singularity.

20. How many roots of the equation $z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lie in the right half plane.

21. Evaluate the integral

$$\int_0^{\pi} \frac{d\theta}{a + \cos\theta}, a > 1$$

by the method of Residues.

22. Suppose that $f_n(z)$ is analytic in a region Ω_n and that the sequence $\{f_n(z)\}$ converges to a limit function $f(z)$ in a region Ω, uniformly on every compact subset of Ω. Prove that $f(z)$ is analytic on Ω.

23. Prove that any two bases of the same module are connected by a unimodular transformation.

24. Show that any elliptic function with periods w_1, w_2 can be written as

$$C \prod_{k=1}^{n} \frac{\sigma(z - a_k)}{\sigma(z - b_k)}$$

where C is a constant.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries 4 *weightage.*

25. If the function $f(z)$ is analytic on a rectangle R, then prove that

$$\int_{2R} f(z)\, dz = 0.$$

26. State and prove the lemma of Schwarz.

27. Describe the Laurent series development.

28. For the Weierstrass $\wp$ -function $\wp(z)$, prove that

$$\wp(z+u) = -\wp(z) - \wp(u) + \frac{1}{4}\left(\frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)}\right).$$

(2 × 4 = 8 weightage)