

C 42027

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Name.....

Reg. No.....

FOURTH SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY)
EXAMINATION, APRIL 2023

(CBCSS)

Mathematics

MTH 4E 09—DIFFERENTIAL GEOMETRY

(2019 Admission onwards)

Time : Three Hours

Maximum : 30 Weightage

Part A*Answer all questions.**Each question has weightage 1.*

1. Sketch level sets of the function $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$ at heights 0, and 4.
2. Sketch the following vector field on \mathbb{R}^2 : $X(p) = (p, X(p))$ where $X(x_1, x_2) = (x_2, x_1)$.
3. Define Gauss map. Illustrate with an example.
4. Define geodesic. Show that geodesics have constant speed.
5. Define Levi-Civita parallel vector field. Show that if X and Y are Levi-Civita parallel vector fields along α , then $X \cdot Y$ is constant along α .
6. Find the normal curvature of $-x_1^2 + x_2^2 + x_3^2 = 1$ at a point on the surface in the direction of v .
7. Define parametrized n -surface in \mathbb{R}^{n+k} , $k \geq 0$.
8. Define (i) Differential 1-form ; and (ii) Exact 1-form.

(8 × 1 = 8 weightage)

Turn over

MODULE III

15. Let V be a finite dimensional vector space with dot product and let $L : V \rightarrow V$ be a self-adjoint linear transformation on V . Let $S = \{v \in V : v \cdot v = 1\}$ and define $f : S \rightarrow \mathbb{R}$ by $f(v) = L(v) \cdot v$. Suppose f is stationary at $v_0 \in S$. Then prove that $L(v_0) = f(v_0)v_0$.
16. Find the Gaussian curvature $K : S \rightarrow \mathbb{R}$ where S is the hyperboloid

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 1.$$

17. State and prove inverse function theorem for n -surface.

(6 × 2 = 12 weightage)

Part C

Answer two questions.

Each question has weightage 5.

18. (a) Let U be an open subset in \mathbb{R}^{n+1} and let $f : U \rightarrow \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f , and let $c = f(p)$. Then prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^\perp$.
- (b) Prove that the gradient of f at $p \in f^{-1}(c)$ is orthogonal to $f^{-1}(c)$ at p .
19. Let C be an oriented plane curve. Then prove that there exists a global parametrization of C if and only if C is connected.
20. Let S be an n -surface in \mathbb{R}^{n+1} and let $p \in S$, and let $v \in S_p$. Then prove that there exists an open interval I containing 0 and a geodesic $\alpha : I \rightarrow S$ such that :
- $\alpha(0) = p$ and $\dot{\alpha}(0) = v$.
 - If $\beta : \tilde{I} \rightarrow S$ is any other geodesic in S with $\beta(0) = p$ and $\dot{\beta}(0) = v$, then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$.

Turn over

21. (a) Let S be an oriented n -surface in \mathbb{R}^{n+1} and let $p \in S$. Let Z be any non-zero normal vector field on S such that $N = \frac{Z}{\|Z\|}$ and let $\{v_1, \dots, v_n\}$ be any basis for S_p . Then prove that

$$K(p) = (-1)^n \det \begin{pmatrix} \nabla_{v_1} Z \\ \vdots \\ \nabla_{v_n} Z \\ Z(p) \end{pmatrix} / \|Z(p)\|^n \det \begin{pmatrix} v_1 \\ \vdots \\ v_n \\ Z(p) \end{pmatrix}.$$

- (b) Let S_1 be an n -surface in \mathbb{R}^{n+1} and let S_2 be an m -surface in \mathbb{R}^{m+1} . Suppose $\varphi: S_1 \rightarrow \mathbb{R}^{m+1}$ is a smooth map such that $\varphi(S_1) \subset S_2$. Show that $d\varphi = T(S_1) \rightarrow T(S_2)$.

(2 × 5 = 10 weightage)