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Reg. No..................................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2018

(CUCSS—PG)

Mathematics

MT 3C 12—MULTIVARIATE CALCULUS AND GEOMETRY

(2016 Syllabus Year)

Time : Three Hours Maximum : 36 Weightage

Part A (Short Answer Questions)

Answer **all** *questions* (1–14).

Each question has 1 *weightage.*

1. Let A be a linear transformation from the vector space $\mathbb{R}^n$ to $\mathbb{R}^m$. Prove that $\| A \| \leq \infty$.

2. Let A, B be linear transformation from the vector space $\mathbb{R}^n$ to $\mathbb{R}^m$. Prove that $\| A + B \| \leq \| A \| + \| B \|$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by :

$$f(x, y) = \begin{cases} \dfrac{x^3}{x^2 + y^2} & \text{if } (x, y) \neq 0 \\ 0 & \text{if } (x, y) = 0 \end{cases}.$$

If u is any unit vector in $\mathbb{R}^2$, then show that the directional derivative $(D_u f)(0, 0)$ exists.

4. Define contraction. Give an example of contraction.

5. Let [A] be a square matrix. If [A] has two columns equal, then prove that det $[A] = 0$.

6. Find a parametrization of the parabola $y = x^2$.

7. Find the Cartesian equation of the parametrized curve $\gamma(t) = (\cos^2 t, \sin^2 t)$.

Turn over

8. Show that the curve $\gamma(t)=\left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)$ is of unit speed.

9. Compute the curvature of the curve $\gamma(t)=(t, \cosh t)$.

10. Prove that the curve $\gamma(t)=\left(t, t^2, t^3\right), t \in \mathbb{R}$ is a regular curve.

11. Define smooth surface and give an example of it.

12. Calculate the first fundamental form of the surface $\sigma(u, v)=\left(u, v, u^2+v^2\right)$.

13. Prove that the second fundamental form of a plane is zero.

14. Calculate the mean curvature of the surface $\sigma(u, v)=(u+v, u-v, uv)$ at the point $(2, 0, 1)$.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *from the following ten questions.* (15–24).
Each question has weightage 2.

15. Prove that a linear operator A on a finite dimensional vector space X is one-one if and only if the range of A is all of X.

16. Let Ω be the set of invertible linear operators on $\mathbb{R}^n$. Prove that Ω is an open subset of $L\left(\mathbb{R}^n\right)$ and the mapping $A \rightarrow A^{-1}$ is continuous on Ω.

17. Let f maps a convex open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$ and let f be differentiable in E. If $f'(x)=0$ for all $x \in E$, then prove that f is a constant.

18. If the tangent vector of a parametrized curve is constant, then prove that that the image of the curve is part of a straight line.

19. Calculate the arc length of the catenary $\gamma(t)=(t, \cosh t)$ starting at the point $(0,1)$.

20. If γ is a unit speed curve, then prove that $\ddot{\gamma}$ is zero or perpendicular to $\dot{\gamma}$.

21. Let S_1, S_2 be surfaces and let $f: S_1 \rightarrow S_2$ be a diffeomorphism. If σ_1 is an allowable surface patch of S_1, the prove that $f \circ \sigma_1$ is an allowable surface patch of S_2.

22. Show that the application of an isometry of $\mathbb{R}^3$ to a surface does not change its first fundamental form.

23. Prove that the normal curvature of any curve on a sphere of radius r is $\pm \frac{1}{r}$.

24. If k_1 and k_2 are the principal curvatures of a surface, then prove that the mean and Gaussian curvatures are given by

$$H = \frac{1}{2}(k_1 + k_2), \text{ and } K = k_1 k_2.$$

(7 × 2 = 14 weightage)

Part C

Answer any **two** *from the following ten questions* (25–28).
Each question has weightage 4.

25. Let f map an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$ and let f be differentiable at $x \in E$. Prove that the partial derivatives $(D_j f_i)(x)$ exist and

$$f'(x)\, e_j = \sum_{i=1}^{m} (D_j f_i)(x)\, u_i,$$

where $1 \le j \le n$ and $\{e_1, e_2, \ldots\ldots\ldots, e_n\}, \{u_1, u_2, \ldots\ldots\ldots, u_m\}$ are standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

26. Prove that a parametrised curve has a unit-speed parametrization if and only if it is regular.

27. Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Prove that its torsion is given by

$$\tau = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{\| \dot{\gamma} \times \ddot{\gamma} \|^2},$$

where × denotes the cross product and dot denotes $\frac{d}{dt}$.

28. Let S be a connected surface of which every point is an umbilic. Prove that S is an open subset of a plane or sphere.

(2 × 4 = 8 weightage)