D 42552 (Pages : 3) Name.................................

Reg. No.............................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2018

(CUCSS)

Mathematics

Group IV : MT 4E 15—WAVELET THEORY

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*
Each question has weightage 1.

1. Let $E_m \in l^2(\mathbb{Z}_N)$ be defined by $E_m(n) = e^{2\pi imn/N}$. Show that $\langle E_m, E_k \rangle = 0$ if $m \neq k$.

2. Let $z = (1, \sqrt{2}, 1/\sqrt{2}, 1) \in l^2(\mathbb{Z}_N)$. Find $\hat{z}(1)$.

3. Give the circulant matrix whose first row is 1, 0, 1.

4. Find $w * z$ where $w = (1, 1, 0, 0)$ and $z = (0, 0, 1, 1)$.

5. Let $\hat{u} = (\sqrt{2}, 0, 0, 1)$ and $\hat{v} = (0, \sqrt{2}, 1, 0)$. Find A (1).

6. For $z = (1, 0, 0, 1) \in l^2(\mathbb{Z}_4)$ find $U_z \in l^2(\mathbb{Z}_8)$.

7. Describe the elements of the space $l^2(\mathbb{Z})$.

8. Let $z(j) = \begin{cases} 1 \text{ if } j \text{ is even and } |j| \leq 4 \\ 0 \text{ otherwise.} \end{cases}$ Verify whether $\langle z, z \rangle = 1$ in $l^2(\mathbb{Z})$.

9. Find the Fourier coefficient c_1 for $f(\theta) = \begin{cases} 1 \text{ if } -\pi \leq \theta < 0 \\ 0 \text{ if } 0 \leq \theta < \pi. \end{cases}$

10. Let $z \in l^2(\mathbb{Z})$ and $\overline{z}(n) = z(-n)$. Show that $\overline{z}$ is in $l^2(\mathbb{Z})$.

11. Let $u \in l^2(\mathbb{Z})$ be given by $u(n) = \begin{cases} 1/\sqrt{2} \text{ if } n = 0 \text{ or } n = 1 \\ 0 \text{ otherwise.} \end{cases}$ Show that $\{R_{2k}u : k = 0, 1, 2\}$ is orthonormal in $l^2(\mathbb{Z})$.

Turn over

12. Let $f \in L^2(\mathbb{R})$ be defined by $f(x) = \begin{cases} 1 \text{ if } 0 \le x \le 1 \\ 0 \text{ otherwise} \end{cases}$. Find $\| f \|$.

13. Show that if $f_n \to f$ in $L^2(R)$ then $\breve{f}_n \to \breve{f}$.

14. Define discrete wavelet transform in $l^2(\mathbb{R})$.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question has weightage 2.

15. Let $\hat{z}$ be the discrete Fourier transform of $z \in l^2(\mathbb{Z}_N)$. Show that $\hat{z}(m+N) = \hat{z}(m)$ for all m.

16. Let A be an N × N circulant matrix and b the first column of A. Show that $Az = b * z$ where $*$ denotes convolution.

17. Show that for $w \in l^2(\mathbb{Z}_4)$, if $|\hat{w}(n)| = 1$ for all n then $\{R_k w : k = 0, 1, 2, 3\}$ is an orthonormal basis of $l^2(\mathbb{Z}_4)$.

18. Let $u \in l^2(\mathbb{Z}_4)$ and $v(k) = \overline{u(1-k)}$ for all k. Show that u and v are orthogonal.

19. Show by an example that $U(D(z)) \ne z$ for $z \in l^2(\mathbb{Z}_N)$.

20. Let $\{a_j\}_{j \in \mathbb{Z}}$ be an orthonormal set in a Hilbert space H and $z \in l^2(\mathbb{Z})$. Show that :

$$\left\| \sum_{j \in \mathbb{Z}} z(j)\, a_j \right\|^2 = \sum |z(j)|^2.$$

21. Let H be a Hilbert space and $T : H \to H$ be a bounded linear operator on H. Show that if :

$\sum_{n \in \mathbb{Z}} x_n$ and $\sum T(x_n)$ are convergent in H then $T\left(\sum x_n\right) = \sum T(x_n)$.

22. Let $w \in l^1(\mathbb{Z})$. Show that if $|\hat{w}(\theta)|^2 + |\hat{w}(\theta+\pi)|^2 = 2$ for all $\theta \in [0, \pi)$ then $\{R_{2k} w\}$ is orthonormal.

23. Give an example of a function $f \in L^1(\mathbb{R})$ such that 0 is a Lebesgue point of f and f is not continuous at 0.

24. Show that if $f, g \in L^2(\mathbb{R})$ then $\langle \breve{f}, \breve{g} \rangle = 1/2\pi \langle f, g \rangle$ where $\breve{f}$ denotes the inverse Fourier transform of f.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question has weightage 4.

25. (a) Describe the discrete Fourier transform and inverse Fourier transform on $l^2(\mathbb{Z}_N)$.

    (b) For $z \in l^2(\mathbb{Z}_N)$ let $\hat{z}$ denote the Fourier transform and $\check{z}$ denote the inverse Fourier transform of z. Show that $(\check{\hat{z}})(n) = z(n)$ for all $n = 0,1,2,\ldots,N-1$.

26. Let N = 2M and $z \in l^2(\mathbb{Z}_N)$. Let $u(k) = z(2k)$ for $k = 0, 1, 2, \ldots, M - 1$ and $v(k) = z(2k+1)$ for $k = 0,1,2\ldots, M-1$. Express the Fourier transform $\hat{z}$ in terms of the Fourier transforms $\hat{u}$ and $\hat{v}$. Prove the result.

27. (a) Describe the trigonometric system in $L^2([-\pi, \pi])$.

    (b) Show that the trigonometric system is a complete orthonormal system in $L^2([-\pi, \pi])$.

28. (a) Define convolution $f * g$ in $L^2(\mathbb{R})$.

    (b) For $f \in L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ prove that :

    (i) $f * g \in L^2(\mathbb{R})$.

    (ii) $\| f * g \| \leq \| f \| \cdot \| g \|_1$.

(2 × 4 = 8 weightage)