C 61904 (Pages : 2) Name..............................

Reg. No..............................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2019

(CUCSS–PG)

Mathematics

MT 4E 02—ALGEBRAIC NUMBER THEORY

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

Section A

Answer **all** *questions.*

Each question carries a weightage of 1.

1. Find the order of the group G/H where G is free abelian with Z-basis x, y, z and H is generated by $41x + 32y - 999z$, $16y + 3z$, $2y + 111z$.
2. Find all monomorphisms $\sigma : Q(\sqrt[3]{7}) \to C$.
3. Show that an algebraic integer is a rational number iff it is a rational integer.
4. Compute integral basis and discriminant of $Q(\sqrt[3]{2})$.
5. Let $K = Q(\xi)$ where $\xi = e^{\frac{2\pi i}{5}}$. Calculate $N_K(\alpha)$ and $T_K(\alpha)$ for $\alpha = \xi + \xi^2$.
6. Determine the group of units of $Z[i]$.
7. Let D be an integral domain and $x, y \in D$. Show that x and y are associates iff x/y and y/x.
8. Show that every Euclidean domain is a principal ideal domain.
9. Find all fractional ideals of $Z[\sqrt{-1}]$.
10. Show that if $\alpha = \langle a \rangle$ is a principal ideal in the ring of integers of a number field of degree n, then $N(\alpha) = |N(a)|$.
11. Show that if L is an n-dimensional lattice in R^n, then $\frac{R^n}{L}$ is isomorphic to the n-dimensional torus T^n.
12. Let $K = Q(\theta)$ where $\theta \in R$ and $\theta^3 = 3$. Determine the σ-map.
13. Let D be the ring of integers of a number field K of degree n. Show that the factorization in D is unique iff the class group H has order 1.
14. Let $K = Q(\sqrt{3})$. Factorize the principal ideal <5> is the ring of integers of K.

(14 × 1 = 14 weightage)

Turn over

Section B

*Answer any **seven** questions.*
Each question carries a weightage of 2.

15. Let G be a free abelian group of rank n with basis $\{x_1, x_2, \ldots, x_n\}$. Suppose (a_{ij}) is an $n \times n$ matrix with integer entries. Show that the elements $y_i = \sum_j a_{ij} x_j$ form a basis of G iff (a_{ij}) is unimodular.

16. Show that the algebraic integers form a subring of the field of algebraic numbers.

17. Let $K = Q(\theta)$ be a number field where θ has minimum polynomial P of degree n. Show that the Q-basis $\{1, \theta, \ldots, \theta^{n-1}\}$ has discriminant $\Delta[1, \theta, \ldots, \theta^{n-1}] = (-1)^{\frac{n(n-1)}{2}} . N(DP(\theta))$, where DP is the formal derivative of P.

18. Let d be a square free rational integer. Show that the ring of integers of $Q(\sqrt{\alpha})$ are $Z[\sqrt{d}]$ if $d \not\equiv 1 \pmod 4$.

19. Show that an integral domain D is noetherian if and only if D satisfies the ascending chain condition.

20. Show that the factorization into irreducible is not unique in the ring of integers of $Q(\sqrt{-21})$.

21. Prove that the ideals $p = \langle 2, 1+\sqrt{-5} \rangle$ and $q = \langle 3, 1+\sqrt{-5} \rangle$ are prime ideals and determine pq.

22. State and prove Minkowski's theorem.

23. With usual notations, prove that Q-linearly independent elements of K map under σ to R-linearly independent elements of L^{st}.

24. Find all square free integers d in $-10 < d < 10$ such that the class number of $Q(\sqrt{d})$ is 1.

(7 × 2 = 14 weightage)

Section C

*Answer any **two** questions.*
Each question carries a weightage of 4.

25. Let R be a ring. Show that every symmetric polynomial in $R[t_1, t_2, \ldots, t_n]$ is expressible as a polynomial with coefficients in R in the elementary symmetric polynomials $s_1, s_2, \ldots, s_n$ and vice versa.

26. Show that the ring of integers of $Q(\sqrt{5})$ is norm Euclidean.

27. Factorize the ideal <14> into prime ideals in $Z[\sqrt{-10}]$.

28. State and prove Kummer's lemma.

(2 × 4 = 8 weightage)