D 70970 (Pages : 3) Name..............................

Reg. No..............................

THIRD SEMESTER M.Sc. DEGREE (REGULAR) EXAMINATION NOVEMBER 2019

(CUCSS)

Mathematics

MT 3C 12—MULTIVARIATE CALCULUS AND GEOMETRY

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*

Each question carries 1 *weightage.*

1. Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then prove that dimension of X is less than equal to r.
2. Prove that $d(\mathrm{A}, \mathrm{B}) = ||\mathrm{A} - \mathrm{B}||$ is a metric on the set of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.
3. If A is a linear transformation from the vector space $\mathbb{R}^n$ to the vector space $\mathbb{R}^m$ and if $x \in \mathbb{R}^n$, then prove that $\mathrm{A}'(x)$, the derivative of A at x, is A.
4. State inverse function theorem.
5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = 2x^3 + 3y^2$. Find the gradient of f at $(1, -1)$.
6. What is meant by a closed curve ? Give an example.
7. Prove that the parametrisation of a given level curve is not unique.
8. If the tangent vector of a parametrised curve is constant, then prove that the image of the curve is a part of a straight line.
9. Show that the curve $\gamma(t) = \left(\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t\right)$ has unit-speed.
10. Calculate the arc-length of the logarithmic spiral $\gamma(t) = \left(e^t \cos t, e^t \sin t\right)$ starting at the point (1, 0).
11. What is meant by a quadric ? Give an example.
12. Calculate the Gaussian and mean curvatures of the surface $\sigma(u,v) = (u+v, u-v, uv)$ at the point (2, 0, 1).
13. What is meant by Weingarten map ?
14. Prove that any geodesic has constant speed.

(14 × 1 = 14 weightage)

Turn over

Part B

Answer any **seven** *questions.*
Each question carries 2 weightage.

15. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator, and suppose $\varepsilon = \{e_1, \dots, e_n\}$ and $\mathcal{U} = \{u_1, \dots, u_n\}$ are bases in $\mathbb{R}^n$. Prove that $\det([A]_\varepsilon) = \det([A]_u)$ where $[A]_\varepsilon, [A]_u$ denote the matrices of A with respect to $\varepsilon, \mathcal{U}$ respectively.

16. Prove that a linear operator A on a finite dimensional vector space X is one-to-one if and only if the range of A is all of X.

17. Let Ω be the set of all invertible linear operators on $\mathbb{R}^n$. If $A \in \Omega, B \in L(\mathbb{R}^n)$ and $\|B - A\| \cdot \|A^{-1}\| < 1$, then prove that $B \in \Omega$.

18. If X is a complete metric space, and if ϕ is a contraction of X into X, then prove that there exists unique $x \in X$ such that $\phi(x) = x$.

19. Let A be a linear operator from $\mathbb{R}^{n+m}$ to $\mathbb{R}^n$. Suppose the map A_x defined by $A_x h = A(h, 0)$ for $h \in \mathbb{R}^n$ is invertible. Prove that given $k \in \mathbb{R}^m$ there is a unique $h \in \mathbb{R}^n$ such that $A(h, k) = 0$.

20. Prove that any regular plane curve whose curvature is a positive constant is part of a circle.

21. Show that the level surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where a, b and c are non-zero constants, is a smooth surface.

22. Find the first and second fundamental forms of the surface $\sigma(u,v) = (u - v, u + v, u^2 + v^2)$.

23. Let $\sigma(u, u)$ be a surface patch with first and second fundamental forms $E du^2 + 2F du dv + G dv^2$ and $L du^2 + 2M du dv + N dv^2$, respectively. Prove that the Gaussian curvature $K = \frac{LN - M^2}{EG - F^2}$ and the mean curvature H is $\frac{LG - 2MF + NE}{2(EG - F^2)}$.

24. Prove that a curve on a surface is a geodesic if and only if its geodesic curvature is zero everywhere.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries 4 weightage.

25. Suppose f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Prove that f is continuously differentiable if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \le i \le m, 1 \le j \le n$.

26. Prove that a parametrised curve has a unit-speed reparametrisation if and only if it is regular.

27. Define a surface in $\mathbb{R}^3$. Is the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ a surface. Justify your answer.

28. Determine the geodesics on the unit sphere S^2 by solving the geodesic equations.

(2 × 4 = 8 weightage)