D 31178 (Pages : 3) Name.......................................

Reg. No..................................

THIRD SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, NOVEMBER 2022

[November 2021 session for SDE/Private Students]

(CBCSS)

Mathematics

MTH 3C 12—COMPLEX ANALYSIS

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each question has weightage 1.

1. If the function f defined on a domain G is differentiable at a point a in G then prove that f is continuous at a.

2. Prove that $|\exp z| = \exp(\operatorname{Re} z)$.

3. Prove that $u(x, y) = \log\left(x^2 + y^2\right)$ is harmonic on $G = \mathbb{C} - \{0\}$.

4. Let γ be the closed polygon $[1-i, 1+i, -1+i, -1-i, 1-i]$. Find $\int_\gamma \frac{1}{z+2}\,dz$.

5. Establish Cauchy's Estimate.

6. Evaluate $\int_C \frac{2z^2 + z}{z^2 - 1}\,dz$ where C is $|z| = 1$.

7. If G is an open set which is a-star shaped and if γ_0 is a curve which is constantly equal to a then prove that every closed rectifiable curve in G is homo topic to γ_0.

8. Find the singularities of $f(z) = \frac{\sin z}{z}$ and identify the type of singularities.

(8 × 1 = 8 weightage)

Turn over

Part B

Answer any **six** *questions choosing* **two** *from each unit.*
Each question has weightage **2.**

Unit I

9. Prove that $f(z) = |z|^2 = x^2 + y^2$ has a derivative only at the origin.

10. If u is a real-valued function defined on a region then prove that u has a harmonic conjugate if u is harmonic.

11. Define Cross ratio. Prove that cross ratio remains invarient under Mobius transformation.

Unit II

12. Let $\gamma : [a, b] \to \mathrm{R}$ be non-decreasing. Show that γ is of bounded variation and $\mathrm{V}(\gamma) = \gamma(b) - \gamma(a)$.

13. If $\gamma : [0, 1] \to \mathrm{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$ then prove that $\frac{1}{2\pi i}\int_{\gamma} \frac{dz}{z-a}$ is an integer.

14. State and prove fundamental theorem of algebra.

Unit III

15. Find the image of $\{z : \mathrm{Re}\, z < 0, |\mathrm{Im}\, z| < \pi\}$ under the exponential function.

16. If f has an essential singularity at $z = a$ then prove that for every $\delta > 0$; $f[ann(a; 0; \delta)] = \mathrm{C}$.

17. Prove that the function $f : [a, b] \to \mathrm{R}$ is convex iff the set $\mathrm{A} = \{(x, y) : a \le x \le b \text{ and } f(x) \le y\}$ is convex.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question has weightage 5.

18. If for a given power series $\sum_{n=0}^{\infty} a_n (z-a)^n$ the number $R, 0 \le R \le \infty$ is defined by

$\frac{1}{R} = \limsup |a_n|^{\frac{1}{n}}$ then prove the following :

(a) If $|z-a| < R$, the series converges absolutely.

(b) If $|z-a| > R$, , the terms of the series become unbounded and so the series diverges.

(c) If $0 < r < R$ then the series converges uniformly on $\{z : |z| \le r\}$.

19. State and prove open mapping theorem.

20. Show that if $f : C \to C$ is a continuous function such that f is analytic off $[-1, 1]$ then f is an entire function.

21. (i) If f is analytic in a region G and a is a point in G with $|f(a)| \ge |f(z)|$ for all z in G then prove that f must be a constant function.

(ii) If G is a bounded open set in C and suppose f is a continuous function on G^- which is analytic in G then prove that $\max\{|f(z)| : z \in G^-\} = \max\{|f(z)| : z \in \partial G\}$.

(2 × 5 = 10 weightage)