

SIXTH SEMESTER B.Sc. DEGREE EXAMINATION, MARCH 2019

(CUCBCSS)

Mathematics

MAT 6B 09—REAL ANALYSIS

Time : Three Hours

Maximum : 120 Marks

Section A

*Answer all the twelve questions.**Each question carries 1 mark.*

1. Give an example of a real valued function with set of real numbers as domain which is nowhere continuous.
2. Define uniform continuous function.
3. State Maximum-Minimum Theorem.
4. Define a tagged partition of a closed and bounded interval in \mathbb{R} .
5. Let $f(x) = x^2$ for $x \in [0, 4]$. Calculate the Riemann sum corresponding to the partition $P = (0, 1, 2, 4)$ with the tags at the right end points of the subintervals.
6. State Fundamental Theorem of Calculus (Second form).
7. Define uniform norm of a bounded function.
8. Give an example to show that pointwise convergence of sequence of functions need not implies uniform convergence.
9. State Cauchy Criterion for the uniform convergence of a series of functions.
10. Define improper integral of the second kind.
11. Show that $B(m, n) = B(n, m)$.
12. Define Gamma function.

(12 × 1 = 12 marks)

Section B

*Answer any ten out of fourteen questions.**Each question carries 4 marks.*

13. Let I be a closed and bounded interval and let $f : I \rightarrow \mathbb{R}$ be continuous of I . If $k \in \mathbb{R}$ is any number satisfying $\inf f(I) \leq k \leq \sup f(I)$, then show that there exists a number $c \in I$ such that $f(c) = k$.

Turn over

14. Show that the equation $x = \cos x$ has a solution in the interval $[0, \pi]$.
15. If $f : A \rightarrow \mathbb{R}$ is uniformly continuous on a subset A of \mathbb{R} and if (x_n) is a Cauchy sequence in A , then show that $(f(x_n))$ is a Cauchy sequence in \mathbb{R} .
16. If f and g are uniformly continuous on subset A of \mathbb{R} , show that $f + g$ is uniformly continuous on A .
17. Consider the function h defined by $h(x) = x + 1$ for $x \in [0, 1]$ rational and $h(x) = 0$ for $x \in [0, 1]$ irrational. Show that h is not Riemann integrable.
18. State and prove the Mean Value Theorem for Integrals.
19. State Lebesgue's Integrability Criterion. Using this discuss the Riemann integrability of any step function on $[a, b]$.
20. Let F and G be differentiable on $[a, b]$ and let $f = F'$ and $g = G'$ belongs to $R[a, b]$, then show that
- $$\int_a^b f G = [FG]_a^b - \int_a^b F g.$$
21. Show that $\lim(x/(x+n)) = 0$ for all $x \in \mathbb{R}, x \geq 0$. Also show that the convergence is not uniform on the interval $[0, \infty)$.
22. Discuss the convergence and uniform convergence of $\sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}$.
23. Show that $\int_1^{\infty} \frac{1}{x^{4/3}} dx$ converges.
24. Show that $\int_0^1 \frac{\sin x}{\sqrt{x}} dx$ converges.
25. Show that $\Gamma(1/2) = \sqrt{\pi}$.

(10 × 4 = 40 marks)

Section C

Answer any **six** out of nine questions.

Each question carries 7 marks.

26. State and prove Boundedness Theorem.
27. If $f(x) = x$ and $g(x) = \sin x$, show that both f and g are uniformly continuous of \mathbb{R} , but that their product fg is not uniformly continuous on \mathbb{R} .
28. If $f \rightarrow \mathbb{R}$ is monotone on $[a, b]$, then show that $f \in \mathbb{R}[a, b]$.

29. State and prove Fundamental Theorem of Calculus (First Form).
30. Let (f_n) be a sequence of bounded functions on $A \in \mathbb{R}$. Then show that this sequence converges uniformly on A to a bounded function f if and only if for each $\epsilon > 0$ there is a number $H(\epsilon)$ in \mathbb{N} such that for all $m, n \geq H(\epsilon)$ then $\|f_m - f_n\| \leq \epsilon$.
31. Show that the uniform limit of a sequence of continuous real valued functions on a subset of real numbers is continuous.
32. Show that the improper integral $\int_0^1 \frac{1}{x} dx$ diverges.
33. Let f be a non-increasing function on $[1, \infty)$ such that $f(x) \geq 0$ ($1 \leq x < \infty$). Then show that $\sum_1^\infty f(n)$ will diverge if $\int_1^\infty f(x) dx$ diverges.
34. Derive a relation between Beta and Gamma function.

(6 × 7 = 42 marks)

Section D

Answer any **two** out of three questions.

Each question carries 13 marks.

35. Show that a continuous function on a closed and bounded interval I can be approximated arbitrarily closely by step function.
36. Let $f: [a, b] \rightarrow \mathbb{R}$ and let $c \in (a, b)$. Then show that $f \in \mathbb{R}[a, b]$ if and only if its restriction to $[a, c]$ and $[c, b]$ are both Riemann integrable. In this case show that $\int_a^b f = \int_a^c f + \int_c^b f$.
37. (a) Let $f_n(x) := 1/(1+x)^n$ for $x \in [0, 1]$. Find the pointwise limit f of the sequence f_n on $[0, 1]$. Does f_n converge uniformly to f on $[0, 1]$?
- (b) Let (c_n) be a decreasing sequence of positive numbers. If $\sum c_n \sin nx$ is uniformly convergent, then show that $\lim (nc_n) = 0$.

(2 × 13 = 26 marks)